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Evolution of solitary waves and undular bores in shallow-water flows over a gradual slope with bottom friction

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This paper considers the propagation of shallow-water solitary and nonlinear periodic waves over a gradual slope with bottom friction in the framework of a variable-coefficient Korteweg–de Vries equation. We use the Whitham averaging method, using a recent development of this theory for perturbed integrable equations. This general approach enables us not only to improve known results on the adiabatic evolution of isolated solitary waves and periodic wave trains in the presence of variable topography and bottom friction, modelled by the Chezy law, but also, importantly, to study the effects of these factors on the propagation of undular bores, which are essentially unsteady in the system under consideration. In particular, it is shown that the combined action of variable topography and bottom friction generally imposes certain global restrictions on the undular bore propagation so that the evolution of the leading solitary wave can be substantially different from that of an isolated solitary wave with the same initial amplitude. This non-local effect is due to nonlinear wave interactions within the undular bore and can lead to an additional solitary wave amplitude growth, which cannot be predicted in the framework of the traditional adiabatic approach to the propagation of solitary waves in slowly varying media.

1. Introduction

There have been many studies of the propagation of water waves over a slope, sometimes also subject to the effects of bottom friction. Many of these works have considered linear waves, or have been numerical simulations in the framework of various nonlinear long-wave model equations. Our interest here is in the propagation of weakly nonlinear long water waves over a slope, simultaneously subject to bottom friction, a combination apparently first considered by Miles (1983*a*, *b*), albeit for the special case of a single solitary wave, or a periodic wavetrain. An appropriate model equation for this scenario is the variable-coefficient perturbed Korteweg–de Vries (KdV) equation (see Grimshaw 1981; Johnson 1973*a*, *b*),

$$A_t + cA_x + \frac{c_x}{2}A + \frac{3c}{2h}AA_x + \frac{ch^2}{6}A_{xxx} = -C_D\frac{c}{h^2}|A|A. \tag{1.1}$$

Here $A(x,t)$ is the free surface elevation above the undisturbed depth $h(x)$ and $c(x) = \sqrt{gh(x)}$ is the linear long-wave phase speed. The bottom friction term on the right-hand side is represented by the Chezy law, modelling a turbulent boundary layer. Here C_D is a non-dimensional drag coefficient, often assumed to have a value around 0.01 (Miles 1983*a*, *b*). Other forms of friction could be used (see, for instance, Grimshaw, Pelinovsky & Talipova 2003) but the Chezy law seems to be the most appropriate for water waves in a shallow depth. In (1.1) the first two terms on the left-hand side are the dominant terms, and by themselves describe the propagation of a linear long wave with speed c. The remaining terms on the left-hand side represent, respectively, the effect of varying depth, weakly nonlinear effects and weak linear dispersion. The equation is derived using the usual KdV balance in which the linear dispersion, represented by $\partial^2/\partial x^2$, is balanced by nonlinearity, represented by A. Here we have added to this balance weak inhomogeneity so that c_x/c scales as $h^2\partial^3/\partial x^3$, and weak friction so that C_D scales with $h\partial/\partial x$. Within this basic balance of terms, we can cast (1.1) into the asymptotically equivalent form

$$A_\tau + \frac{h_\tau}{4h}A + \frac{3}{2h}AA_X + \frac{h}{6g}A_{XXX} = -C_D\frac{g^{1/2}}{h^{3/2}}|A|A, \tag{1.2}$$

where

$$\tau = \int_0^x \frac{\mathrm{d}x'}{c(x')}, \qquad X = \tau - t. \tag{1.3}$$

Here we have $h = h(x(\tau))$, explicitly dependent on the variable τ which describes evolution along the path of the wave.

The governing equation (1.2) can be cast into several equivalent forms. That most commonly used is the variable-coefficient KdV equation, obtained here by putting

$$B = (gh)^{1/4}A, \tag{1.4}$$

so that

$$B_\tau + \frac{3}{2g^{1/4}h^{5/4}}BB_X + \frac{h}{6g}B_{XXX} = -C_D\frac{g^{1/4}}{h^{7/4}}|B|B. \tag{1.5}$$

This form shows that, in the absence of friction term, i.e. when $C_D \equiv 0$, equation (1.2) has two integrals of motion with the densities proportional to $h^{1/4}A$ and $h^{1/2}A^2$. These are often referred to as laws for the conservation of 'mass' and 'momentum'. However, these densities do not necessarily correspond to the corresponding physical entities. Indeed, to leading order, the 'momentum' density is proportional to the wave action flux, while the 'mass' density differs slightly from the actual mass density. This latter issue has been explored by Miles (1979), where it was shown that the difference is smaller than the error incurred in the derivation of (1.4), and is due to reflected waves.

Our main concern in this paper is with the behaviour of an undular bore over a slope in the presence of bottom friction, using the perturbed KdV equation (1.2), where we were originally motivated by the possibility that the behaviour of a tsunami approaching the shore might be modelled in this way. The undular bore solution to the unperturbed KdV equation can be constructed using the well-known Gurevich–Pitaevskii (GP) (1974) approach (see also Fornberg & Whitham 1978). In this approach, the undular bore is represented as a modulated nonlinear periodic wavetrain. The main feature of this unsteady undular bore is the presence of a solitary wave (which is the limiting wave form of the periodic cnoidal wave) at its leading edge. The original initial-value problem for the KdV equation is then replaced by a certain boundary-value problem for the associated modulation Whitham equations. We note, however, that so far, the simplest, '(x/t)'-similarity solutions of the modulation equations have been used for the modelling of undular bores in various contexts (see

Grimshaw & Smyth 1986, Smyth 1987 or Apel 2003, for instance). These solutions, while effectively describing many features of undular bores, are degenerate and fail to capture, even qualitatively, some important effects associated with non-self-similar modulation dynamics. In particular, in the classical GP solution for the resolution of an initial jump in the unperturbed KdV equation, the amplitude of the lead solitary wave in the undular bore is constant (twice the value of the initial jump). On the other hand, the modulation solution for the undular bore evolving from a general monotonically decreasing initial profile shows that the lead solitary wave amplitude in fact grows with time (Gurevich, Krylov & Mazur 1989; Gurevich, Krylov & El 1992; Kamchatnov 2000). As we shall see, the very possibility of such variations in the modulated solutions of the unperturbed KdV equation has a very important fluid dynamics implication: *in a general setting, the undular bore lead solitary wave cannot be treated as an individual KdV solitary wave but rather represents a part of the global nonlinear wave structure*. In other words, while at every particular moment of time the lead solitary wave has the spatial profile of the familiar KdV soliton, generally, the temporal dependence of its amplitude cannot be obtained in the framework of single solitary wave perturbation theory.

In the unperturbed KdV equation, the growth of the lead solitary wave amplitude is caused by the spatial inhomogeneity of the initial data. Here, however, the presence of a perturbation due to topography and/or friction serves as an alternative and/or additional cause for variation of the lead solitary wave amplitude. Thus, in the present case, the variation in the amplitude will have two components (which generally, of course, cannot be separated because of the nonlinear nature of the problem); one is local, described by the adiabatic perturbation theory for a single solitary wave, and the other one is non-local, which in principle requires the study of the full modulation solution. Depending on the relative values of the small parameters associated with the slope, friction and spatial non-uniformity of the initial modulations, we can take into account only one of these components, or a combination of them.

The structure of the paper is as follows. First, in § 2, we reformulate the basic model (1.1) as a constant-coefficient KdV equation perturbed by terms representing topography and friction. Then we derive in § 3 the associated perturbed Whitham modulation equations using methods recently developed by Kamchatnov (2004). Next, in § 4, this Whitham system is integrated in the solitary-wave limit. Our purpose here is primarily to obtain the equation of a multiple characteristic, which defines the leading edge of a shoaling undular bore in the case when the modulations due to the combined action of the slope and bottom friction are small compared to the existing spatial modulations due to non-uniformity of the initial data. As a by-product of this integration, we reproduce and extend the known results on the adiabatic variation of a single solitary wave (Miles 1983*a*, *b*). Then, in § 5, we carry out an analogous study of a cnoidal wave, propagating over a gradual slope and subject to friction, a case studied previously by Miles (1983*b*) but under the restriction of zero mean flow, which is removed here. Finally, in § 6 we study the effects of a gradual slope and bottom friction on the front of an undular bore which represents a modulated cnoidal wave transforming into a system of weakly interacting solitons near its leading edge.

2. Problem formulation

For the purpose of the present paper it is convenient to recast (1.2) into the standard KdV equation form with constant coefficients, modified by certain perturbation

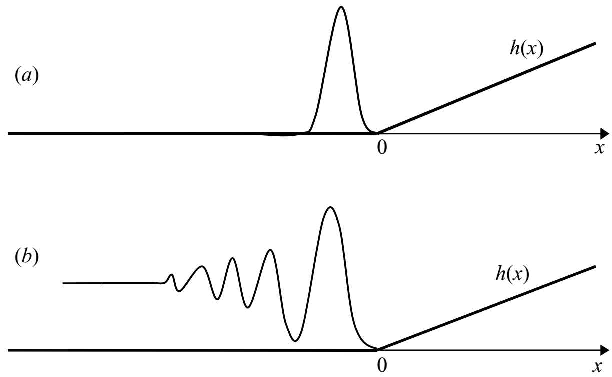

FIGURE 1. Isolated solitary wave (a) and undular bore (b) entering the variable topography/bottom friction region.

terms. Thus we introduce the new variables

$$U = \frac{3g}{2h^2}A, \qquad T = \frac{1}{6g}\int_0^{\tau} h\,\mathrm{d}\tau = \frac{1}{6g^{3/2}}\int_0^{x}\sqrt{h(x)}\,\mathrm{d}x. \tag{2.1}$$

so that

$$U_T + 6UU_X + U_{XXX} = R = F(T)U - G(T)|U|U, \tag{2.2}$$

where

$$F(T) = -\frac{9h_T}{4h}, \qquad G(T) = 4C_D\frac{g^{1/2}}{h^{1/2}}. \tag{2.3}$$

In this form, the governing equation (2.2) has the structure of the integrable KdV equation on the left-hand side, while the separate effects of the varying depth and the bottom friction are represented by the two terms on the right-hand side. This structure enables us to use the general theory developed in Kamchatnov (2004) for perturbed integrable systems.

For much of the subsequent discussion, it is useful to assume that $h(x)=$ constant, $C_D=0$ for $x<0$ in the original equation (1.1), which corresponds to $F(T)=G(T)=0$ for $T<0$ in (2.2). We shall also assume that $A=0$ for $x>0$ at $t=0$, which corresponds to $U=0$ for $X>0$ on $X=\tau(T)$ (see (2.1)). Then we shall propose two types of initial-value problem for (1.1), and correspondingly for (2.2).

(a) Let a solitary wave of a given amplitude a_0 initially propagating over a flat bottom without friction (i.e. a soliton described by an unperturbed KdV equation), enter the variable topography and bottom friction region at $t=0$, $x=0$ (figure 1a).

(b) Let an undular bore of a given intensity propagate over a flat bottom without friction (the corresponding solution of the unperturbed KdV equation will be discussed in § 5). Let the lead solitary wave of this undular bore have the same amplitude a_0 and enter the variable topography and bottom friction region at $t=0$, $x=0$ (figure 1b).

In particular, we shall be interested in the comparison of the slow evolution of these two, initially identical, solitary waves in the two different problems described above. The expected essential difference in the evolution is due to the fact that the lead solitary wave in the undular bore is generally not independent of the remaining part of the bore and can exhibit features that cannot be captured by a local perturbation analysis. The well-known example of such a behaviour, when a solitary wave is constrained by the condition of being a part of a global nonlinear wave structure, is provided by the undular bore solution of the KdV–Burgers (KdV–B) equation

$$u_t + 6uu_x + u_{xxx} = \mu u_{xx}, \qquad \mu \ll 1. \tag{2.4}$$

Indeed, the undular bore solution of the KdV–B equation (2.4) is known to have a solitary wave at its leading edge (see Johnson 1970; Gurevich & Pitaevskii 1987; Avilov, Krichever & Novikov 1987) and this solitary wave: (*a*) is asymptotically close to a soliton solution of the unperturbed KdV equation; and (*b*) has the amplitude, say a_0, that is constant in time. At the same time, it is clear that if one takes an isolated KdV soliton of the same amplitude a_0 as initial data for the KdV–Burgers equation, it would damp with time due to dissipation. The physical explanation of such a drastic difference in the behaviour of an isolated soliton and a lead solitary wave in the undular bore for the same weakly dissipative KdV–B equation is that the action of weak dissipation on an expanding undular bore is twofold: on the one hand, the dissipation tends to decrease the amplitude of the wave locally but, on the other hand, it 'squeezes' the undular bore so that the interaction (i.e. momentum exchange) between separate solitons within the bore becomes stronger than in the absence of dissipation and this acts as the amplitude-increasing factor. The additional momentum is extracted from the upstream flow with a greater depth (see Benjamin & Lighthill 1954). As a result, in the case of the KdV–B equation, an equilibrium non-zero value for the lead solitary wave amplitude in the undular bore is established. Of course, for other types of dissipation, a stationary value of the lead soliton amplitude would not necessarily exist, but in general, due to the expected increase of the soliton interactions near the leading edge, the amplitude of the lead soliton of the undular bore would decay more slowly than that of an isolated soliton. Indeed, the presence here of variable topography as well can result in an additional 'non-local' amplitude growth.

While the problem (*a*) can be solved using traditional perturbation analysis for a single solitary wave, which leads to an ordinary differential equation along the solitary wave path (see Miles 1983*a*, *b*), the undular bore evolution problem (*b*) requires a more general approach which can be developed on the basis of Whitham's modulation theory leading to a system of three nonlinear hyperbolic partial differential equations of the first order. Since the Whitham method, being equivalent to a nonlinear multiple scale perturbation procedure, contains the adiabatic theory of slow evolution of a single solitary wave as a particular (albeit singular) limit, it is instructive for the purposes of this paper to treat both problems (*a*) and (*b*) using the general Whitham theory.

3. Modulation equations

The original Whitham method (Whitham 1965, 1974) was developed for conservative constant-coefficient nonlinear dispersive equations and is based on the averaging of appropriate conservation laws of the original system over the period of a single-phase periodic travelling wave solution. The resulting system of quasi-linear equations describes the slow evolution of the modulations (i.e. of the mean value, the wavenumber, the amplitude, etc.) of the periodic travelling wave. Here, that approach is extended to the perturbed KdV equation (2.1) following the general approach of Kamchatnov (2004), which extends earlier results for certain specific cases (see Gurevich & Pitaevskii 1987, 1991, Avilov, Krichever & Novikov 1987, and Myint & Grimshaw 1995, for instance).

We suppose that the evolution of the nonlinear wave is adiabatically slow, that is, the wave can be locally represented as a solution of the corresponding unperturbed

KdV equation (i.e. (2.2) with zero on the right-hand side) with its parameters slowly varying with space and time. The one-phase periodic solution of the KdV equation can be written in the form

$$U(X,T) = \lambda_3 - \lambda_1 - \lambda_2 - 2(\lambda_3 - \lambda_2)\mathrm{sn}^2(\sqrt{\lambda_3 - \lambda_1}\,\theta, m) \tag{3.1}$$

where $\mathrm{sn}(y,m)$ is the Jacobi elliptic sine function, $\lambda_1 \leqslant \lambda_2 \leqslant \lambda_3$ are parameters and the phase variable θ and the modulus m are given by

$$\theta = X - VT, \quad V = -2(\lambda_1 + \lambda_2 + \lambda_3), \tag{3.2}$$

$$m = \frac{\lambda_3 - \lambda_2}{\lambda_3 - \lambda_1}, \tag{3.3}$$

$$L = \oint \mathrm{d}\theta = \int_{\lambda_2}^{\lambda_3} \frac{\mathrm{d}\mu}{\sqrt{-P(\mu)}} = \frac{2K(m)}{\sqrt{\lambda_3 - \lambda_1}}, \tag{3.4}$$

where $K(m)$ is the complete elliptic integral of the first kind, L is the 'wavelength' along the X-axis (which is actually a retarded time rather than a true spatial coordinate). Here we have used the representation of the basic ordinary differential equation for the KdV travelling wave solution (3.1) in the form (see Kamchatnov 2000 for a general motivation behind this representation)

$$\frac{\mathrm{d}\mu}{\mathrm{d}\theta} = 2\sqrt{-P(\mu)}, \tag{3.5}$$

where

$$\mu = \tfrac{1}{2}(U + s_1), \quad s_1 = \lambda_1 + \lambda_2 + \lambda_3 \tag{3.6}$$

and

$$P(\mu) = \prod_{i=1}^{3}(\mu - \lambda_i) = \mu^3 - s_1\mu^2 + s_2\mu - s_3, \tag{3.7}$$

that is the solution (3.1) is parameterized by the zeros λ_1, λ_2, λ_3 of the polynomial $P(\mu)$.

In a modulated wave, the parameters $\lambda_1, \lambda_2, \lambda_3$ are allowed to be slow functions of X and T, and their evolution is governed by the Whitham equations. For the unperturbed KdV equation, the evolution of the modulation parameters is due to a spatial non-uniformity of the initial distributions for λ_j, $j = 1, 2, 3$ and the typical spatio-temporal scale of the modulation variations is determined by the scale of the initial data.

In the case of the perturbed KdV equation (2.2), the evolution of the parameters $\lambda_1, \lambda_2, \lambda_3$ is caused not only by their initial spatial non-uniformity, but also by the action of the weak perturbation, so that, generally, at least two independent spatio-temporal scales *for the modulations* can be involved. However, at this point we shall not introduce any scale separation within the modulation theory and derive general perturbed Whitham equations assuming that the typical values of $F(T)$ and $G(T)$ are $O(\partial\lambda_j/\partial T, \partial\lambda_j/\partial X)$ within the modulation theory.

It is instructive to first introduce the Whitham equations for the perturbed KdV equation (2.2) using the traditional approach of averaging the (perturbed) conservation laws. To this end, we introduce the averaging over the period (3.4) of the cnoidal wave (3.1) by

$$\langle \mathfrak{F} \rangle = \frac{1}{L}\oint \mathfrak{F}\,\mathrm{d}\theta = \frac{1}{L}\int_{\lambda_2}^{\lambda_3} \frac{\mathfrak{F}\mathrm{d}\mu}{\sqrt{-P(\mu)}}. \tag{3.8}$$

In particular,

$$\langle U\rangle = 2\langle\mu\rangle - s_1 = 2(\lambda_3-\lambda_1)\frac{E(m)}{K(m)} + \lambda_1 - \lambda_2 - \lambda_3, \tag{3.9}$$

$$\langle U^2\rangle = 8\left[-\frac{s_1}{6}(\lambda_3-\lambda_1)\frac{E(m)}{K(m)} - \frac{1}{3}s_1\lambda_1 + \frac{1}{6}\left(\lambda_1^2 - \lambda_2\lambda_3\right)\right] + s_1^2, \tag{3.10}$$

where $E(m)$ is the complete elliptic integral of the second kind. Now, one represents the KdV equation (2.2) in the form of the perturbed conservation laws

$$\frac{\partial P_j}{\partial T} + \frac{\partial Q_j}{\partial X} = R_j, \quad j = 1, 2, 3, \quad R_j \ll 1, \tag{3.11}$$

where P_j and Q_j are the standard expressions for the conserved densities (Kruskal integrals) and 'fluxes' of the unperturbed KdV equation. Just as in the Whitham (1965) theory for unperturbed dispersive systems, the number of conservation laws required is equal to the number of free parameters in the travelling wave solution, which is three in the present case. Next, one applies the averaging (3.8) to the system (3.11) to obtain (see Dubrovin & Novikov 1989)

$$\frac{\partial\langle P_j\rangle}{\partial T} + \frac{\partial\langle Q_j\rangle}{\partial X} = \langle R_j\rangle, \quad j = 1, 2, 3. \tag{3.12}$$

The system (3.12) describes the slow evolution of the parameters λ_j in the cnoidal wave solution (3.1).

Along with this derived perturbed conservative form of the Whitham equations, we introduce the wave conservation law which is a general condition for the existence of slowly modulated single-phase travelling wave solutions (3.1) (see for instance Whitham 1974) and must be consistent with the modulation system (3.12). This conservation law has the form

$$\frac{\partial k}{\partial T} + \frac{\partial \omega}{\partial X} = 0, \tag{3.13}$$

where

$$k = \frac{2\pi}{L}, \qquad \omega = kV \tag{3.14}$$

are the 'wavenumber' and the 'frequency' respectively (we have put quotation marks here because the actual wavenumber and frequency related to the physical variables x, t are different quantities from those in (3.14), but are related through the transformations (1.3), (2.1)). The wave conservation law (3.13) can be introduced instead of any of three inhomogeneous averaged conservation laws comprising the Whitham system (3.12).

It is known that the Whitham system for the homogeneous constant-coefficient KdV equation can be represented in diagonal (Riemann) form (Whitham 1965, 1974) by an appropriate choice of the three parameters characterizing the periodic travelling wave solution. In fact, in our solution (2.2) the parameters λ_j have already been chosen so that they coincide with the Riemann invariants of the unperturbed KdV modulation system. Introducing them explicitly into the perturbed system (3.12) we obtain (see Kamchatnov 2004)

$$\frac{\partial\lambda_i}{\partial T} + v_i\frac{\partial\lambda_i}{\partial X} = \frac{L}{\partial L/\partial\lambda_i} \times \frac{\langle(2\lambda_i - s_1 - U)R\rangle}{4\prod_{j\neq i}(\lambda_i-\lambda_j)}, \quad i = 1, 2, 3, \tag{3.15}$$

where R is the perturbation term on the right-hand side of the KdV equation (2.2) and

$$v_i = -2\sum \lambda_i + \frac{2L}{\partial L/\partial \lambda_i}, \quad i = 1, 2, 3, \tag{3.16}$$

are the Whitham characteristic velocities corresponding to the unperturbed KdV equation.

It should be noted that the straightforward realization of the above lucid general algorithm for obtaining perturbed modulation system in diagonal form is quite a laborious task. In fact, to derive system (3.15), the so-called finite-gap integration method incorporating the integrable structure of the unperturbed KdV equation has been used. The modulation system (3.15) in a more particular form corresponding to specific choices of the perturbation term was obtained by Myint and Grimshaw (1995) using a multiple-scale perturbation expansion. In that latter setting, the wave conservation law (3.13) is an inherent part of the construction, while in the averaging approach used here, it can be obtained as a consequence of the system (3.15).

To obtain an explicit representation of the Whitham equations for the present case of equation (2.2), we must substitute the perturbation R from the right-hand side of (2.2) and perform the integration (3.8) with U given by (3.1). From now on, we are going to consider only the flows where $U \geqslant 0$ so that the perturbation term assumes the form

$$R(U) = G(T)U - F(T)U^2. \tag{3.17}$$

Substituting (3.17) into (3.15), we obtain, after some detailed calculations (see the Appendix), the perturbed Whitham system in the form

$$\frac{\partial \lambda_i}{\partial T} + v_i \frac{\partial \lambda_i}{\partial X} = \rho_i = C_i[F(T)A_i - G(T)B_i], \quad i = 1, 2, 3, \tag{3.18}$$

where

$$C_1 = \frac{1}{E}, \quad C_2 = \frac{1}{E-(1-m)K}, \quad C_3 = \frac{1}{E-K}; \tag{3.19}$$

$$\left.\begin{aligned} A_1 &= \tfrac{1}{3}(5\lambda_1 - \lambda_2 - \lambda_3)E + \tfrac{2}{3}(\lambda_2 - \lambda_1)K, \\ A_2 &= \tfrac{1}{3}(5\lambda_2 - \lambda_1 - \lambda_3)E - (\lambda_2 - \lambda_1)\left(\tfrac{1}{3} + \frac{\lambda_2}{\lambda_3 - \lambda_1}\right)K, \\ A_3 &= \tfrac{1}{3}(5\lambda_3 - \lambda_1 - \lambda_2)E - \left[\lambda_3 + \tfrac{1}{3}(\lambda_2 - \lambda_1)\right]K; \end{aligned}\right\} \tag{3.20}$$

$$\left.\begin{aligned} B_1 &= \tfrac{1}{15}\left(-27\lambda_1^2 - 7\lambda_2^2 - 7\lambda_3^2 + 2\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 22\lambda_2\lambda_3\right)E \\ &\quad - \tfrac{4}{15}(\lambda_2 - \lambda_1)(3\lambda_1 + \lambda_2 + \lambda_3)K, \\ B_2 &= \tfrac{1}{15}\left(-7\lambda_1^2 - 27\lambda_2^2 - 7\lambda_3^2 + 2\lambda_1\lambda_2 + 22\lambda_1\lambda_3 + 2\lambda_2\lambda_3\right)E \\ &\quad + \tfrac{1}{15}\frac{\lambda_2 - \lambda_1}{\lambda_3 - \lambda_1}\left(7\lambda_1^2 + 15\lambda_2^2 + 11\lambda_3^2 - 6\lambda_1\lambda_2 - 18\lambda_1\lambda_3 + 6\lambda_2\lambda_3\right)K, \\ B_3 &= \tfrac{1}{15}\left(-7\lambda_1^2 - 7\lambda_2^2 - 27\lambda_3^2 + 22\lambda_1\lambda_2 + 2\lambda_1\lambda_3 + 2\lambda_2\lambda_3\right)E \\ &\quad + \tfrac{1}{15}\left(7\lambda_1^2 + 11\lambda_2^2 + 15\lambda_3^2 - 18\lambda_1\lambda_2 - 6\lambda_1\lambda_3 + 6\lambda_2\lambda_3\right)K; \end{aligned}\right\} \tag{3.21}$$

and the characteristic velocities are:

$$\left.\begin{aligned} v_1 &= -2\sum \lambda_i + \frac{4(\lambda_3-\lambda_1)(1-m)K}{E}, \\ v_2 &= -2\sum \lambda_i - \frac{4(\lambda_3-\lambda_2)(1-m)K}{E-(1-m)K}, \\ v_3 &= -2\sum \lambda_i + \frac{4(\lambda_3-\lambda_2)K}{E-K}. \end{aligned}\right\} \tag{3.22}$$

The equations (3.18)–(3.22) provide a general setting for studying the nonlinear modulated wave evolution over variable topography with bottom friction. In the absence of the perturbation terms (i.e. when $F(T)\equiv 0$, $G(T)\equiv 0$), the system (3.18), (3.22) indeed coincides with the original Whitham equations (Whitham 1965) for the integrable KdV dynamics. In that case the variables λ_1, λ_2, λ_3 become Riemann invariants, so in this general (perturbed) case we shall call them Riemann variables.

It is important to study the structure of the perturbed Whitham equations (3.18)–(3.22) in two limiting cases when the underlying cnoidal wave degenerates into (i) a small-amplitude sinusoidal wave (linear limit), when $\lambda_2=\lambda_3$ ($m=0$), and (ii) a solitary wave when $\lambda_2=\lambda_1$ ($m=1$). Since in both these limits the oscillations do not contribute to the mean flow (they are infinitely small in the linear limit and the distance between them becomes infinitely long in the solitary wave limit) one should expect that in both cases one of the Whitham equations will transform into the dispersionless limit of the original perturbed KdV equation (2.2) i.e.

$$U_T + 6UU_X = F(T)U - G(T)U^2. \tag{3.23}$$

Indeed, using formulae (3.18)–(3.22) we obtain for $m = 0$:

$$\left.\begin{aligned} &\lambda_2 = \lambda_3, \\ &\frac{\partial \lambda_1}{\partial T} - 6\lambda_1 \frac{\partial \lambda_1}{\partial X} = \lambda_1 F + \lambda_1^2 G, \\ &\frac{\partial \lambda_3}{\partial T} + (6\lambda_1 - 12\lambda_3)\frac{\partial \lambda_3}{\partial X} = \lambda_1 F + \lambda_1^2 G. \end{aligned}\right\} \tag{3.24}$$

Similarly, for $m = 1$, one has

$$\left.\begin{aligned} &\lambda_2 = \lambda_1, \\ &\frac{\partial \lambda_1}{\partial T} - (4\lambda_1 + 2\lambda_3)\frac{\partial \lambda_1}{\partial X} = \frac{1}{3}(4\lambda_1 - \lambda_3)F + \frac{1}{15}\left(7\lambda_3^2 - 24\lambda_1\lambda_3 + 32\lambda_1^2\right)G, \\ &\frac{\partial \lambda_3}{\partial T} - 6\lambda_3 \frac{\partial \lambda_3}{\partial X} = \lambda_3 F + \lambda_3^2 G \end{aligned}\right\} \tag{3.25}$$

We see that, in both cases, one of the Riemann variables (taken with inverted sign) coincides with the solution of the dispersionless equation (3.23) (recall that in the derivation of the Whitham equations we assumed $U \geqslant 0$ everywhere), namely $U=\langle U\rangle=-\lambda_1$ when $\lambda_2=\lambda_3$ ($m=0$) and $U=\langle U\rangle=-\lambda_3$ when $\lambda_2=\lambda_1$ ($m=1$).

To conclude this section, we present expressions for the physical wave parameters such as the surface elevation wave amplitude a, mean elevation $\langle A\rangle$, speed and wavenumber in terms of the modulation solution $\lambda_j(X,T)$. Using (2.1) and (3.1) we obtain for the wave amplitude (peak to trough) and the mean elevation

$$a = \frac{4h^2}{3g}(\lambda_3 - \lambda_2), \qquad \langle A\rangle = \frac{2h^2}{3g}\langle U\rangle, \tag{3.26}$$

where the dependence of $\langle U\rangle$ on $\lambda_j(X,T)$, $j=1,2,3$ is given by (3.9) and $X=X(x,t)$, $T=T(x,t)$ by (1.3), (2.1). In order to obtain the physical wavenumber κ and the frequency Ω we first note that the phase function $\theta(X,T)$ defined in (3.2) is replaced by a more general expression defined so that $k=\theta_X$ and $kV=-\theta_T$ are the 'wavenumber' and 'frequency' in the $X-T$ coordinate system. Then we define the physical phase function $\Theta(x,t)=\theta(X,T)$ so that we get

$$\kappa=\Theta_x,\qquad \Omega=-\Theta_t. \tag{3.27}$$

It now follows that

$$\kappa=\frac{k}{c}\left(1-\frac{hV}{6g}\right),\quad \Omega=k,\quad \frac{\Omega}{\kappa}=\frac{c}{1-hV/6g}. \tag{3.28}$$

Note that the physical frequency is the 'wavenumber' in the $X-T$ coordinate system, and that the physical phase speed is Ω/κ. Since the validity of the KdV model (1.1) requires *inter alia* that the wave be right-going, it follows from this expression that the modulation solution remains valid only when $hV<6g$. Of course, the validity of (1.1) also requires that the amplitude remains small, and this would normally also ensure that V remains small.

4. Modulation solution in the solitary wave limit

In this section, we shall integrate the perturbed modulation system (3.18) along the multiple characteristic corresponding to the merging of two Riemann variables λ_2 and λ_1. As we shall see later, this characteristic specifies the motion of the leading edge of the shoaling undular bore in the case when the perturbations due to variable topography and bottom friction can be considered as small compared with the existing spatial modulations within the bore. At the same time, as the case $\lambda_2=\lambda_1$ (i.e. $m=1$) corresponds to the solitary wave limit in the travelling wave solution (3.1), our results here are expected to be consistent with the results from the traditional perturbation approach to the adiabatic variation of a solitary wave due to topography and bottom friction (see Miles 1983*a*, *b*).

In the limit $m\to 1$ the periodic solution (3.1) of the KdV equation goes over to its solitary wave solution

$$U(X,T)=U_0\,\mathrm{sech}^2[\sqrt{\lambda_3-\lambda_1}(X-V_sT)]-\lambda_3, \tag{4.1}$$

where

$$U_0=2(\lambda_3-\lambda_1),\qquad V_s=-(4\lambda_1+2\lambda_3) \tag{4.2}$$

are the solitary wave amplitude and 'velocity' respectively. The solution (4.1) depends on two parameters λ_1 and λ_3 whose adiabatic slow evolution is governed by the reduced modulation system (3.25). It is important that the second equation in this system is decoupled from the first one. Hence, evolution of the pedestal $-\lambda_3$ on which the solitary wave rides, can be found from the solution of this dispersionless equation by the method of characteristics. When $\lambda_3(X,T)$ is known, evolution of the parameter λ_1 can be found from the solution of the first equation (3.25). As a result, we arrive at a complete description of adiabatic slow evolution of the solitary wave parameters taking account of its interaction with the (given) pedestal.

However, it is important to note here that while this description of the adiabatic evolution of a solitary wave is complete as far as the solitary wave itself is concerned, it fails to describe the evolution of a trailing shelf, which is needed to conserve total 'mass' (see, for instance, Johnson 1973*b*, Grimshaw 1979 or Grimshaw 2007). This trailing shelf has a very small amplitude, but a very large length scale, and hence can carry the same order of 'mass' as the solitary wave. But note that the 'momentum' of the trailing shelf is much smaller than that of the solitary wave, whose adiabatic deformation is in fact governed to leading order by conservation of 'momentum', or more precisely, by conservation of wave action flux (strictly speaking, conservation only in the absence of friction).

The situation simplifies if the solitary wave propagates into a region of still water so that there is no pedestal ahead of the wave, that is $\lambda_3 = 0$ in $X > \tau(T)$. But then, since $\lambda_3 = 0$ is an exact solution of the degenerate Whitham system (3.25) for this solitary wave configuration, we can put $\lambda_3 = 0$ both in the solitary wave solution,

$$U(X, T) = -2\lambda_1 \operatorname{sech}^2[\sqrt{-\lambda_1}(X - V_s T)], \quad V_s = -4\lambda_1, \tag{4.3}$$

and in (3.25) for the parameter λ_1 to obtain

$$\frac{\partial \lambda_1}{\partial T} - 4\lambda_1 \frac{\partial \lambda_1}{\partial X} = \frac{4}{3} F \lambda_1 + \frac{32}{15} G \lambda_1^2. \tag{4.4}$$

As we see, the solitary wave moves with the instantaneous velocity

$$\frac{\mathrm{d}X}{\mathrm{d}T} = -4\lambda_1, \tag{4.5}$$

and the parameter λ_1 changes with T along the solitary wave trajectory according to the ordinary differential equation

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}T} = \frac{4}{3} F(T) \lambda_1 + \frac{32}{15} G(T) \lambda_1^2. \tag{4.6}$$

It can be shown that (4.6) is consistent with the equation for the solitary wave half-width $\gamma = \sqrt{-\lambda_1}$ obtained by the traditional perturbation approach (see Grimshaw 1979, for instance).

Next, we rewrite (4.6) in terms of the original independent x-variable. We find from (2.1) that

$$\mathrm{d}T = \left(h^{1/2}/6g^{3/2}\right) \mathrm{d}x, \tag{4.7}$$

$$F = -\frac{27}{2}\left(\frac{g}{h}\right)^{3/2} \frac{\mathrm{d}h}{\mathrm{d}x}, \, G = 4C_D \left(\frac{g}{h}\right)^{1/2}. \tag{4.8}$$

Then substituting these expressions into (4.6) yields the equation

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}x} = -3\frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}\lambda_1 + \frac{64}{45}\frac{C_D}{g}\lambda_1^2, \tag{4.9}$$

which can be easily integrated to give

$$\frac{1}{\lambda_1} = h^3 \left(-C_0 - \frac{64}{45}\frac{C_D}{g} \int_0^x \frac{\mathrm{d}x}{h^3}\right), \tag{4.10}$$

where C_0 is an integration constant and $x = 0$ is a reference point where $h = h_0$. According to (4.3), $U_0 = -2\lambda_1$ is the amplitude of the soliton expressed in terms of variable $U(X, T)$. Returning to the original surface displacement $A(x, t)$ by means of (2.1) and denoting $C_0 = 4/(3ga_0h_0)$, we find the dependence of the surface elevation soliton amplitude $a = (2h^2/3g)U_0$ on x in the form

$$a = a_0 \left(\frac{h_0}{h}\right) \left[1 + \frac{16}{15} C_D a_0 h_0 \int_0^x \frac{\mathrm{d}x}{h^3}\right]^{-1}, \tag{4.11}$$

where a_0 is the solitary wave amplitude at $x=0$. We note that for $C_D=0$ this reduces to the classical Boussinesq (1872) result $a \sim h^{-1}$, while for $h=h_0$ it reduces to the well-known algebraic decay law $a \sim 1/(1+\text{constant } x)$ due to Chezy friction. Miles (1983*a*, *b*) obtained this expression for a linear depth variation, although we note that there is a factor of 2 difference from (4.11) (in Miles 1983*a, b*, the factor $16C_D/15$ is $8C_D/15$). The trajectory of the soliton can be now found from (4.5) and (4.10):

$$X=\int_0^x \frac{\mathrm{d}x}{\sqrt{gh}}-t=\frac{a_0h_0}{2\sqrt{g}}\int_0^x \mathrm{d}x'h^{-5/2}(x')\left[1+\frac{16}{15}C_Da_0h_0\int_0^{x'}\frac{\mathrm{d}x}{h^3(x)}\right]^{-1}. \tag{4.12}$$

This expression determines implicitly the dependence of x on t along the solitary wave path and provides the desired equation for the multiple characteristic of the modulation system for the case $m=1$.

It is instructive to derive an explicit expression for the solitary wave speed by computing the derivative $\mathrm{d}x/\mathrm{d}t$ from (4.12), or more simply, directly from (3.28),

$$v_s=\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{c}{1-a/2h}. \tag{4.13}$$

The formula (4.13) yields the restriction for the relative amplitude $\gamma=a/h<2$ which is clearly beyond the applicability of the KdV approximation (wave breaking occurs already at $\gamma=0.7$; see Whitham 1974). In the frictionless case ($C_D=0$), equation (4.11) gives $a/h=a_0h_0/h^2$, and so the expression (4.13) for the speed must fail as $h\to 0$. It is interesting to note that this failure of the KdV model as $h\to 0$ due to appearance of infinite (and further negative!) solitary wave speeds is not apparent from the expression (4.11) for the solitary wave amplitude, and the implication is that the model cannot be continued as $h\to 0$. Curiously this restriction of the KdV model seems never to have been noticed before in spite of numerous works on this subject. Note that taking account of bottom friction leads to a more complicated formula for the solitary wave speed as a function of h but the qualitative result remains the same.

It is straightforward to show from (4.9) or (4.11) that

$$\frac{a_x}{a}=-\frac{h_x}{h}-\frac{16}{15}\frac{C_Da_0h_0}{h^3}\left[1+\frac{16}{15}C_Da_0h_0\int_0^x\frac{\mathrm{d}x}{h^3}\right]^{-1}. \tag{4.14}$$

It follows immediately that for a wave advancing into increasing depth ($h_x>0$), the amplitude decreases due to a combination of increasing depth and bottom friction. However, for a wave advancing into decreasing depth, there is a tendency to increase the amplitude due to the depth decrease, but to decrease the amplitude due to bottom friction. Hence, whether or not the amplitude increases is determined by which of these effects is larger, and this in turn is determined by the slope, the depth, and the consolidated drag parameter C_Da_0/h_0.

To illustrate, let us consider the bottom topography in the form

$$h(x)=h_0^{1-\alpha}(h_0-\delta x)^\alpha,\quad \alpha>0, \tag{4.15}$$

which satisfies the condition $h(0)=h_0$; the parameter δ characterizes the slope of the bottom. In this case the formula (4.11) becomes

$$a=a_0\left(\frac{h_0}{h}\right)\left[1+\frac{16}{15}\frac{C_Da_0}{\delta(3\alpha-1)h_0}\left\{\left(\frac{h_0}{h}\right)^{(3\alpha-1)/\alpha}-1\right\}\right]^{-1} \tag{4.16}$$

if $\alpha \neq 1/3$. One can see now that if $\alpha < 1/3$, then the bottom friction term is relatively unimportant due to the smallness of C_D. Of course, for this case we again recover the Boussinesq result, now slightly modified,

$$a \approx a_0 \frac{h_0}{h} \left[1 + \frac{16}{15} \frac{C_D a_0}{\delta(1-3\alpha)h_0^2} \right]^{-1}, \quad 0 < \alpha < \tfrac{1}{3}, \quad h \ll h_0. \tag{4.17}$$

Of course, this result is impractical in the KdV context as the KdV approximation used here requires the ratio a/h to remain small.

If $\alpha > 1/3$ we obtain the asymptotic formula

$$a \approx \frac{15(3\alpha-1)\delta}{16C_D} h_0 \left(\frac{h_0}{h} \right)^{1/\alpha - 2}, \quad h \ll h_0, \tag{4.18}$$

which is independent of the initial amplitude a_0. This expression is consistent with the small-amplitude KdV approximation as long as $(3\alpha - 1)\delta/C_D$ is of order unity. Simple inspection of (4.18) shows that the solitary wave amplitude

(i) increases as $h \to 0$ if $1/3 < \alpha < 1/2$,
(ii) is constant as $h \to 0$ if $\alpha = 1/2$,
(iii) decreases as $h \to 0$ if $\alpha > 1/2$.

Thus for $1/3 < \alpha < 1/2$, as for the case $\alpha < 1/3$, the amplitude will increase as the depth decreases, in spite of the presence of (sufficiently small) friction. However, for $\alpha > 1/3$, even although there is usually some initial growth in the amplitude, eventually even small bottom friction will take effect and the amplitude decreases to zero. We note that if $\alpha = 1/3$ then the integral $\int_0^x h^{-3} \mathrm{d}x$ in (4.11) diverges logarithmically as $h \to 0$, which just slightly modifies the result (4.18) for $h \ll h_0$ and implies growth of the amplitude $\propto \ln h/h$ as $h \to 0$.

Of particular interest is the case $\alpha = 1$. In that case formula (4.16) becomes

$$a = a_0 \left(\frac{h_0}{h} \right) \left[1 + \frac{8}{15} \frac{C_D a_0}{\delta h_0} \left\{ \left(\frac{h_0}{h} \right)^2 - 1 \right\} \right]^{-1}. \tag{4.19}$$

$$a \approx \frac{15}{8} \frac{\delta}{C_D} h, \qquad h \ll h_0 \tag{4.20}$$

These expressions (4.19), and (4.20) were obtained by Miles (1983*a*, *b*) using wave energy conservation (as above, note, however, that in Miles 1983*a*, *b* the numerical coefficient is 15/4 rather than 15/8). Thus, these results obtained from the Whitham theory are indeed consistent, to leading order, with the traditional perturbation approach for a slowly varying solitary wave.

5. Adiabatic deformation of a cnoidal wave

Next we consider a modulated cnoidal wave (3.1) in the special case when the modulation does not depend on X. While this case is, strictly speaking, impractical, as it assumes there is an infinitely long wavetrain, it can nevertheless provide some useful insights into the qualitative effects of gradual slope and friction on undular bores which are locally represented as cnoidal waves. In the absence of friction, the slow dependence of the cnoidal wave parameters on T was obtained by Ostrovsky & Pelinovsky (1970, 1975) and Miles (1979) (see also Grimshaw 2007), assuming that the surface displacement had a zero mean (i.e. $\langle U \rangle = 0$), while the effects of friction were taken into account by Miles (1983*b*) using the same zero-mean displacement assumption. However, this assumption is inconsistent with our aim to study undular bores where the value of $\langle U\rangle$ is essentially non-zero. Hence, we need to develop a more general theory enabling us to take into account variations in all the parameters in the cnoidal wave. Such a general setting is provided by the modulation system (3.18).

Thus we consider the case when the Riemann variables in (3.18) do not depend on the variable X so that the general Whitham equations become ordinary differential equations in T, which can be conveniently reformulated in terms of the original spatial x-coordinate using the relationship (4.7):

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}x} = C_i\left[-\frac{9}{4}\frac{1}{h}\frac{dh}{dx}A_i - \frac{2C_D}{3g}B_i\right], \qquad i = 1, 2, 3, \tag{5.1}$$

where all variables are defined above in § 3, see (3.19), (3.20), (3.21). This system can be readily solved numerically. But it is instructive, however, to indicate first some general properties of the solution.

First, the solution to the system (5.1) must have the property of conservation of 'wavelength' L (or 'wavenumber' $k = 2\pi/L$)

$$L = \frac{2K(m)}{\sqrt{\lambda_3 - \lambda_1}} = \text{constant}. \tag{5.2}$$

Indeed, the wave conservation law (3.13) in the absence of X-dependence assumes the form

$$\frac{\partial k}{\partial T} = 0, \tag{5.3}$$

which yields (5.2). Thus the system of three equations (5.1) can be reduced to two equations.

Next, applying Whitham averaging directly to (2.2) yields

$$\frac{\mathrm{d}M}{\mathrm{d}x} = -\frac{9}{4}\frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}M - \frac{2C_D}{3g}\tilde{P}, \quad M = \langle U\rangle, \quad \tilde{P} = \langle |U|U\rangle, \tag{5.4}$$

$$\frac{\mathrm{d}P}{\mathrm{d}x} = -\frac{9}{2}\frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}x}P - \frac{4C_D}{3g}\tilde{Q}, \quad P = \langle U^2\rangle, \quad \tilde{Q} = \langle |U|^3\rangle. \tag{5.5}$$

The equation set (5.2), (5.4), (5.5) comprises a closed modulation system for three independent modulation parameters, say M, $\tilde{P}$ and m. While this system is not as convenient for further analysis as the system (3.18) in Riemann variables, it does not have a restriction $U > 0$ inherent in (3.18), and allows for some straightforward inferences regarding the possible existence of modulation solutions with zero mean elevation, that is with $M = 0$. Indeed, one can see that the solution with the zero mean is actually not generally permissible when $C_D \neq 0$, a situation overlooked in Miles (1983*b*). Indeed, $M = 0$ immediately then implies that $\tilde{P} = 0$ by (5.4). But then, due to (5.2), we have all three modulation parameters fixed, which is clearly inconsistent with the remaining equation (5.5) (except for the trivial case $M = 0$, $P = 0$, $\tilde{Q} = 0$). However, in the absence of friction, when $C_D = 0$, equation (5.4) uncouples and permits a non-trivial solution with a zero mean. In general, when $C_D = 0$, (5.4) and (5.5) can be easily integrated to give

$$d = Mh^{9/4} = \text{constant}; \quad \sigma = Ph^{9/2} = \text{constant}. \tag{5.6}$$

Then, using (3.9), (3.10), (5.2), one readily gets the formula for the variation of the modulus m, and hence of all the other wave parameters, as a function of h:

$$K^2[2(2-m)EK - 3E^2 - (1-m)K^2] = \left(\frac{4}{3}\right)^5 \frac{(\sigma - d^2)L^4}{h^{9/2}}. \tag{5.7}$$

Formula (5.7) generalizes to the case $M \neq 0$ (i.e. $d \neq 0$) the expressions of Ostrovsky & Pelinovsky (1970, 1975), Miles (1979) and Grimshaw (2006) (note that in Grimshaw 2007, the zero mean restriction is actually not necessary). We note here that, again with $C_D = 0$, equation (1.5) implies conservation of $\langle B \rangle$ and $\langle B^2 \rangle$ (the averaged wave action flux), which, together with (5.2), also yield (5.7).

The physical frequency Ω and wavenumber κ in the modulated periodic wave under study are given by the formula (3.28), and we recall here that $k = 2\pi/L$ is constant; see (5.2). As discussed before, at the end of § 3, we must require that the phase speed stays positive as the wave evolves, and here that requires that the physical wavenumber $\kappa > 0$. Since a/h (and hence $hV/6g$) is supposed to be small within the range of applicability of the KdV equation (1.2) the expression (3.28) implies the behaviour $\kappa \simeq \Omega/\sqrt{gh}$, which of course agrees with the well-known result for linear waves on a sloping beach (see Johnson 1997, for instance). This effect will be slightly attenuated for the nonlinear cnoidal wave, since $Vh/6g > 0$, but the overall effect will be a 'squeezing' of the cnoidal wave, a result important for our further study of undular bores. Next we study numerically the combined effect of slope and friction on a cnoidal wave.

As we have shown, in the presence of Chezy friction $M \neq 0$, and we have also assumed that $U > 0$, which is necessary when we come to study undular bores. Now we use the stationary modulation system (5.1) in Riemann variables, which was derived using this assumption. We solve the coupled ordinary differential equation system (5.1) for the case of a linear slope

$$h(x) = h_0 - \delta x \tag{5.8}$$

with $h_0 = 10$, $\delta = 0.01$, and with the initial conditions

$$\lambda_1 = -0.441, \quad \lambda_2 = 0.147, \quad \lambda_3 = 0.294 \quad \text{at} \quad x = 0, \tag{5.9}$$

which corresponds to a nearly harmonic wave with $m = 0.2$, $a/h_0 = 0.2$, $\langle A \rangle / h_0 \approx 0.3$ at $x = 0$ (see (3.26)). Also we note that for the chosen parameters we have $V = 0$, so at $x = 0$ we have $\kappa = \Omega/\sqrt{gh_0}$ as in linear theory. It is instructive to compare solutions with ($C_D = 0.01$) and without ($C_D = 0$) friction. In figure 2 the dependence of the modulus m on x is shown for both cases. We see that for the frictionless case $m \to 1$ with decrease of depth, i.e. the wave crests assume the shape of solitary waves when one approaches the shoreline. When $C_D \neq 0$ the modulus also grows with decrease of depth but never reaches unity. The dependence on x of the mean surface elevation $\langle A \rangle$ for the cases without and with friction is shown in figure 3. We have checked that the 'wavelength' L (5.2) is constant for both solutions. Also, one can see from figure 3(*b*) that the value $h^{1/4}\langle A \rangle \propto d$ is indeed conserved in the frictionless case but is not constant if friction is present (the same holds true for the value $h^{1/2}\langle A^2 \rangle \propto \sigma$ but we do not present the graph here). Finally, in figure 4 the dependence of the physical elevation wave amplitude a on the spatial coordinate x is shown. One can see that the amplitude adiabatically grows with distance in the frictionless case due to the effect of the slope (without friction) but, not unexpectedly, gradually decreases in the case when bottom friction is present, where the decrease for these parameter settings is comparable in magnitude to the effect of the slope. In both cases the main qualitative changes occur in the wave shape and the wavelength.

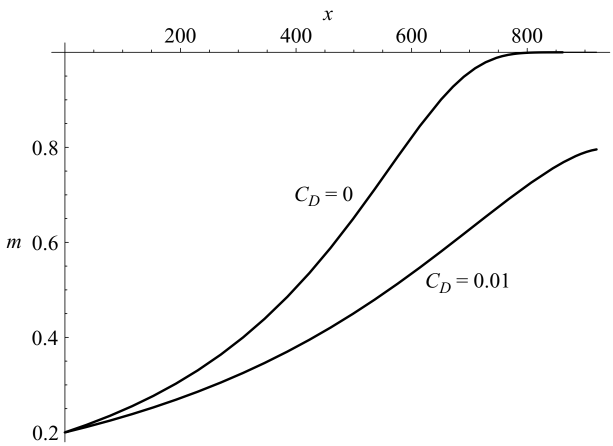

FIGURE 2. Dependence of the modulus m on the physical space coordinate x in the cases without and with bottom friction in the X-independent modulation solution.

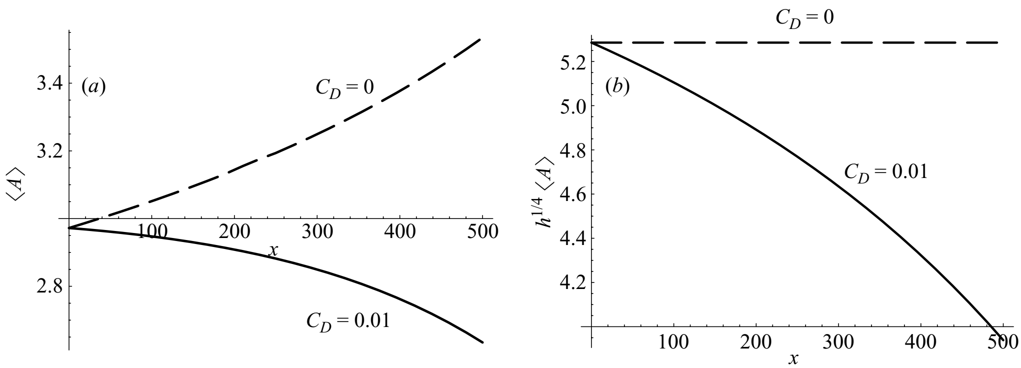

FIGURE 3. (a) Dependence of the mean value $\langle A\rangle$ in the X-independent modulation solution on the physical space coordinate x without (dashed line) and with (solid line) bottom friction; (b) The same but multiplied by the Green's law factor, $h^{1/4}$.

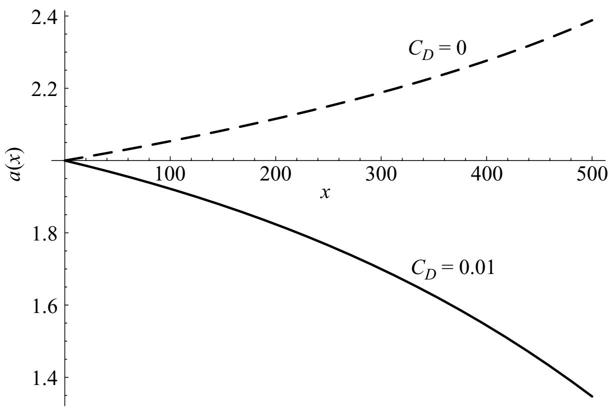

FIGURE 4. Dependence of the surface elevation amplitude a on the space coordinate x. The dashed line corresponds to the frictionless case and the solid line to the case with bottom friction.

Overall, we can infer from these results that the main local effect of a slope and bottom friction on a cnoidal wave, along with the adiabatic amplitude variations, is twofold: a wave with a $m < 1$ at $x = 0$ tends to transform into a sequence of solitary waves as x increases, and at the same time the distance between subsequent wave crests tends to decrease. This is in sharp contrast with the behaviour of modulated cnoidal waves in problems described by the unperturbed KdV equation, where growth of the modulus m is accompanied by an *increase* of the distance between the wave crests. Generally, in the study of behaviour of unsteady undular bores in the presence of a slope and bottom friction we will have to deal with the combination of these two opposite tendencies.

6. Undular bore propagation over variable topography with bottom friction

6.1. *The Gurevich–Pitaevskii problem for the flat-bottom zero-friction case*

We now turn to the problem (*b*) outlined in § 2. We study the evolution of an undular bore developing from an initial surface elevation jump $\Delta > 0$, located at some point $x_0 < 0$. As discussed below, the undular bore will expand with time so that at some $t = t_0$ its lead solitary wave enters the gradual slope region, which begins at $x = 0$ (see figure 1b). We assume that for $x < 0$ one has $h = h_0 = \text{constant}$ and $C_D \equiv 0$. We shall first present a formulation of the Gurevich–Pitaevskii problem for the perturbation-free KdV equation and reproduce the well-known similarity modulation solution describing the evolution of the undular bore until the moment it enters the slope. We emphasize that, although this formulation and, especially, this similarity solution are known very well and have been used by many authors, some of the inferences important for the present application to fluid dynamics have not been widely appreciated, as far as we can discern. Pertinent to our main objective in this paper, we undertake a detailed study of the characteristics of the Whitham modulation system in the vicinity of the leading edge of the undular bore solution, and show that the boundary conditions of Gurevich–Pitaevskii type permit only two possible characteristics configurations, implying two qualitatively different types of the leading solitary wave behaviour. Next, we shall show how this Gurevich–Pitaevskii formulation of the problem applies to the perturbed modulation system in the form (3.18) and finally we will study the effects of the perturbation on the modulations in the vicinity of the leading edge of the undular bore.

In the case of a flat, frictionless bottom the original equation (1.1) becomes the constant-coefficient KdV equation which can be cast into the standard form

$$\eta_\zeta + 6\eta\eta_\xi + \eta_{\xi\xi\xi} = 0 \tag{6.1}$$

by introducing the new variables

$$\eta = \frac{2}{3h_0}A, \quad \xi = \frac{3}{2h_0}(x + x_0 - \sqrt{gh_0}t), \quad \zeta = \frac{9}{16}\sqrt{\frac{g}{h_0}}t, \tag{6.2}$$

where $x_0 < 0$ is an arbitrary constant. In the Gurevich–Pitaevskii (GP) approach, one considers a large-scale initial disturbance $\eta(\xi, 0) = f(\xi)$, in the form of a decreasing profile, $f'(\xi) < 0$ (e.g. a smooth step: $f(\xi) \to 0$ as $\xi \to +\infty$; $f(\xi) \to \eta_0 > 0$ as $\xi \to -\infty$), whose initial evolution until some critical (breaking) time ζ_b can be described by the dispersionless limit of the KdV equation, i.e. by the Hopf equation,

$$\zeta < \zeta_b : \qquad \eta \approx r(\xi, \zeta), \qquad r_\zeta + 6rr_\xi = 0, \qquad r(\xi, 0) = f(\xi). \tag{6.3}$$

The evolution (6.3) leads to wavebreaking of the $r(\xi)$-profile at some $\zeta=\zeta_b$, with the consequence that the dispersive term in the KdV equation then comes into play, and an undular bore forms, which can be locally represented as a single-phase travelling wave. This travelling wave is modulated in such a way that it acquires the form of a solitary wave at the leading edge $\xi=\xi^+(\zeta)$ and gradually degenerates, via the nonlinear cnoidal-wave regime, to a linear wave packet at the trailing edge $\xi=\xi^-(\zeta)$. It is important that this undular bore is essentially unsteady, i.e. the region $\xi^-(\zeta)<\xi<\xi^+(\zeta)$ expands with time ζ.

The single-phase travelling wave solution of the KdV equation (6.1) has the form (*cf.* (3.1))

$$\eta(\xi,\zeta)=r_3-r_1-r_2-2(r_3-r_2)\mathrm{sn}^2(\sqrt{r_3-r_1}\theta,m), \tag{6.4}$$

$$\theta=\xi+2(r_1+r_2+r_3)\zeta, \qquad m=\frac{r_3-r_2}{r_3-r_1}. \tag{6.5}$$

The parameters $r_1\leqslant r_2\leqslant r_3\leqslant 0$ in the undular bore are slowly varying functions of ξ,ζ, whose evolution is governed by the Whitham equations

$$\frac{\partial r_j}{\partial\zeta}+v_j(r_1,r_2,r_3)\frac{\partial r_j}{\partial\xi}=0, \qquad j=1,2,3. \tag{6.6}$$

The characteristic velocities in (6.6) are given by (3.22). We stress that, although analytical expressions (6.4) and (3.1) (as well as (6.6) and the homogeneous version of (3.18)) are identical, they are written for completely different sets of variables, both dependent and independent.

The Riemann invariants $r_j(\xi,\zeta)$ are subject to special matching conditions at the free boundaries, $\xi=\xi^\pm(\zeta)$ defined by the conditions $m=0$ (trailing edge) and $m=1$ (leading edge), formulated in Gurevich & Pitaevskii (1974) (see also Kamchatnov 2000 or El 2005 for a detailed description).

At the trailing (harmonic) edge, where the wave amplitude $a=2(r_3-r_2)$ vanishes and $m=0$, we have

$$\xi=\xi^-(\zeta): \qquad r_2=r_3, \quad -r_1=r. \tag{6.7}$$

At the leading (soliton) edge, where $m=1$, we have

$$\xi=\xi^+(\zeta): \qquad r_2=r_1, \quad -r_3=r. \tag{6.8}$$

In both (6.7) and (6.8), $r(\xi,\zeta)$ is the solution of the Hopf equation (6.3).

The curves $\xi=\xi^\pm(\zeta)$ are defined for the solution of the GP problem (6.6), (6.7), (6.8) by the ordinary differential equations

$$\frac{\mathrm{d}\xi^-}{\mathrm{d}\zeta}=v^-(\xi^-,\zeta), \qquad \frac{\mathrm{d}\xi^+}{\mathrm{d}\zeta}=v^+(\xi^+,\zeta), \tag{6.9}$$

where $v^\pm$ are calculated as the values of double characteristic velocities of the modulation system at the undular bore edges,

$$v^-=v_2(r_1,r_3,r_3)|_{\xi=\xi^-(\zeta)}=v_3(r_1,r_3,r_3)|_{\xi=\xi^-(\zeta)}, \tag{6.10}$$

$$v^+=v_2(r_1,r_1,r_3)|_{\xi=\xi^+(\zeta)}=v_1(r_1,r_1,r_3)|_{\xi=\xi^+(\zeta)} \tag{6.11}$$

These equations (6.9) essentially represent kinematic boundary conditions for the undular bore (see El 2005). Indeed, the double characteristic velocity $v_2(r_1,r_3,r_3)=v_3(r_1,r_3,r_3)$ can be shown to coincide with the linear group velocity of the small-amplitude KdV wavepacket while the double characteristic velocity $v_2(r_1,r_1,r_3)=v_1(r_1,r_1,r_3)$ is the soliton speed.

One might infer from this GP formulation of the problem that, since the leading edge of the undular bore specified by (6.9) and (6.11) is a characteristic of the modulation system, then the value of the double Riemann invariant $r^+ \equiv r_2 = r_1$ is constant. Then, on considering an undular bore propagating into still water, where $r = 0$, one would obtain from the matching condition (6.8) at the leading edge that $r_3|_{\xi=\xi^+} = 0$, and thus the amplitude of the lead solitary wave $a^+ = 2(r_3 - r_1)|_{\xi=\xi^+} = -r^+$ would always be constant as well. However, this contradicts the general physical reasoning that the amplitude of the lead solitary wave should be allowed to change in the case of general initial data. The apparent contradiction is resolved by noting that the leading edge specified by (6.9) and (6.11) can be an *envelope* of the characteristic family, i.e. a caustic, rather than necessarily a regular characteristic, and hence there is no necessity for the double Riemann invariant r^+ to be constant along the curve $\xi = \xi^+(\zeta)$ in the general case. On the other hand, since the leading edge is defined by the condition $m = 1$, the wave form at the leading edge will coincide with the spatial profile of the standard KdV soliton. Thus we arrive at the conclusion that, in general, the amplitude of the leading KdV solitary wave will vary, even in the absence of the perturbation terms. Of course, in the unperturbed KdV equation, such varying solitary waves cannot exist on their own, and require the presence of the rest of the undular bore. We also stress that these variations of the leading solitary wave in the undular bore, as described here, have a completely different physical nature to the variations of the parameters of an individual solitary wave due to small perturbations as described in §4. They are caused by nonlinear wave interactions within the undular bore rather than by a local adiabatic response of the solitary wave to a perturbation induced by topography and friction. Importantly for our study, however, it will transpire that the action of these same perturbation terms on the undular bore can lead to both a local and a non-local response of the leading solitary wave.

6.2. *Undular bore developing from an initial jump*

Next we consider the simplest solution of the modulation system, which describes an undular bore developing from an initial discontinuity placed at the point $x = -x_0$. In $(\eta; \xi, \zeta)$-variables we have the initial conditions

$$\eta(\xi, 0) = \Delta \quad \text{for} \quad \xi < 0\,; \qquad \eta(\xi, 0) = 0 \quad \text{for} \quad \xi > 0, \tag{6.12}$$

where $\Delta > 0$ is a constant. Then, on using (6.3), the initial conditions (6.12) are readily translated into the free-boundary matching conditions (6.7) and (6.8) for the Riemann invariants. Because of the absence of a length scale in this problem, the corresponding solution of the modulation system must depend on the self-similar variable $\tau = \xi/\zeta$ alone, which reduces the modulation system to the ordinary differential equations

$$(v_i - \tau)\frac{\mathrm{d}r_i}{\mathrm{d}\tau} = 0, \qquad i = 1, 2, 3. \tag{6.13}$$

The boundary conditions for (6.13) follow from the matching conditions (6.7) and (6.8) using the initial condition (6.12):

$$\begin{aligned} \tau = \tau^- : &\qquad r_2 = r_3, \quad r_1 = -\Delta, \\ \tau = \tau^+ : &\qquad r_2 = r_1, \quad r_3 = 0, \end{aligned} \tag{6.14}$$

where $\tau^\pm$ are self-similar coordinates (speeds) of the leading and trailing edges, $\xi^\pm = \tau^\pm \zeta$. Taking into account the inequality $r_1 \leqslant r_2 \leqslant r_3$, one obtains the well-known modulation solution of Gurevich & Pitaevskii (1974) (see also Fornberg & Whitham

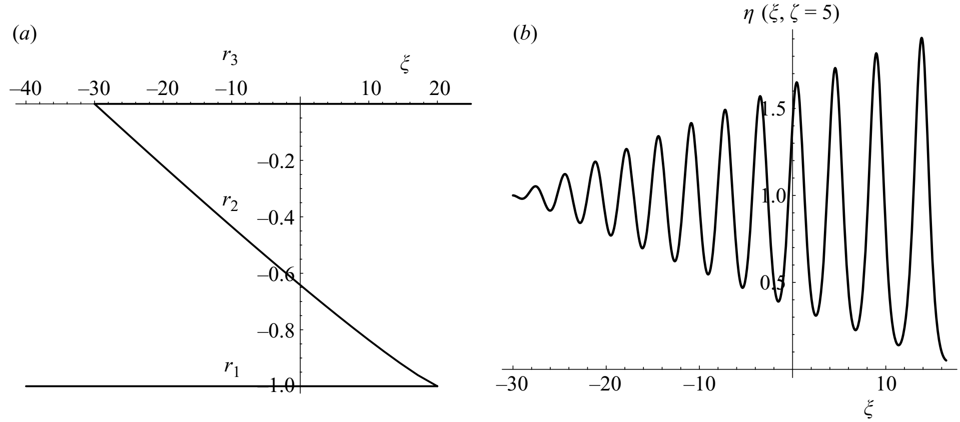

FIGURE 5. (a) Behaviour of Riemann invariants in the similarity modulation solution for the flat-bottom zero-friction case. (b) The corresponding undular bore profile $\eta(\xi)$.

1978) in the form

$$r_1 = -\Delta, \quad r_3 = 0, \quad r_2 = -m\Delta, \tag{6.15}$$

$$\frac{\xi}{\zeta} = v_2(-\Delta, -m\Delta, 0) = 2\Delta\left[(1+m) - \frac{2m(1-m)K(m)}{E(m)-(1-m)K(m)}\right]. \tag{6.16}$$

This modulation solution (6.15), (6.16) (see figure 5a) represents the replacement, due to averaging over the oscillations, of the unphysical formal three-valued solution of the dispersionless KdV equation (i.e. of the Hopf equation) which would have taken place in the absence of the dispersive regularisation by the undular bore. We see that (6.16) describes an expansion fan in the characteristic (ξ, ζ)-plane and thus is a global solution. Substituting (6.15), (6.16) into the travelling wave solution (6.4), we obtain the asymptotic wave form of the undular bore (see figure 5b), which then can be readily represented in terms of the original physical variables using the relationships (6.2).

The equations of the trailing and leading edges of the undular bore are determined from (6.16) by putting $m=0$ and $m=1$ respectively:

$$\frac{\xi^-}{\zeta} = \tau^- = v_2(-\Delta, 0, 0) = -6\Delta, \qquad \frac{\xi^+}{\zeta} = \tau^+ = v_2(-\Delta, -\Delta, 0) = 4\Delta. \tag{6.17}$$

The leading solitary wave amplitude is $\eta_0 = 2(r_3 - r_1) = 2\Delta$, which is exactly twice the height of the initial jump. This corresponds to the amplitude of the surface elevation $a = 3h_0\Delta$ (see (6.2)). Note that, to get the leading solitary wave of the same initial amplitude a_0 as for the separate solitary wave considered in §4, one should use the jump value $\Delta_0 = a_0/3h_0$, which of course is just $2\tilde{\Delta}$, where $\tilde{\Delta} = 3h_0\Delta/2$ is the initial discontinuity in the surface elevation.

6.3. *Structure of the undular bore front*

We are especially interested in the behaviour of the modulation solution (6.15), (6.16) in the vicinity of the leading edge $\xi = \xi^+(\zeta)$. This behaviour is essentially determined by the manner in which the pair of characteristics corresponding to the velocities v_2 and v_1 merge into a multiple eigenvalue v^+ of the modulation system at $\xi = \xi^+(\zeta)$.

First, one can readily infer from the modulation solution (6.15), (6.16) that the phase velocity $c = -2(r_1 + r_2 + r_3) = 2\Delta(1+m) > v_2(-\Delta, -m, 0)$ for $m < 1$ and $c = v_2$ for $m = 1$. Thus, any individual wave crest generated at the trailing edge of the

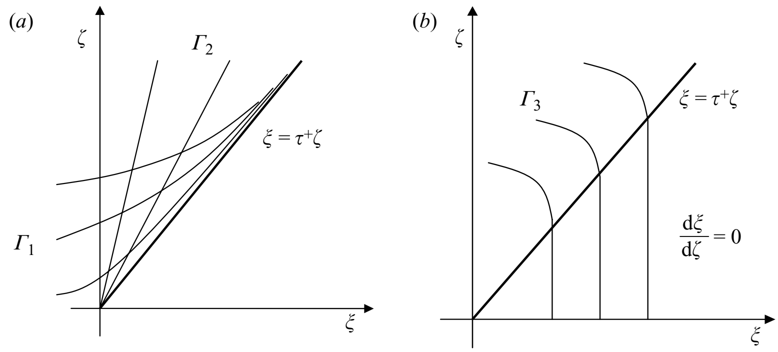

FIGURE 6. Characteristics' behaviour for the similarity modulation solution near the leading edge $\xi^+(\zeta)$: (*a*) families Γ_1: $\mathrm{d}\xi/\mathrm{d}\zeta = v_1$ and Γ_2: $\xi = C_2\zeta$, (*b*) family Γ_3: $\mathrm{d}\xi/\mathrm{d}\zeta = v_3$.

undular bore moves towards the leading edge, i.e. for any crest $m \to 1$ as $\zeta \to \infty$. Thus, for any particular wave crest, except for the very first one, the solitary wave 'status' is achieved only asymptotically as $\zeta \to \infty$.

Without loss of generality we assume in this section that $\Delta = 1$ in (6.15), (6.16). First, as we have already mentioned, the characteristic family Γ_2 : $\mathrm{d}\xi/\mathrm{d}\zeta = v_2$ is an expansion fan in the (ξ, ζ)-plane,

$$\Gamma_2 : \qquad \xi = C_2\zeta, \tag{6.18}$$

parametrized by a constant C_2, $-6 \leqslant C_2 \leqslant 4$. Next, in (6.16) we make an asymptotic expansion of $v_2(-1, -m, 0)$ for small $(1 - m) \ll 1$, to get

$$2(1 - m)\ln(16/(1 - m)) \simeq \tau^+ - \xi/\zeta \tag{6.19}$$

or, with logarithmic accuracy,

$$(\tau^+ - \xi/\zeta) \ll 1 : \qquad 1 - m \simeq \frac{\tau^+ - \xi/\zeta}{2\ln[1/(\tau^+ - \xi/\zeta)]}. \tag{6.20}$$

Next, expanding $v_1(-1, -m, 0)$ for $(1-m) \ll 1$ and using (6.20), we get the asymptotic equation for the characteristics family Γ_1,

$$\frac{\mathrm{d}\xi}{\mathrm{d}\zeta} = v_1 = \tau^+ + (\tau^+ - \xi/\zeta) + O(1 - m), \tag{6.21}$$

which is readily integrated to leading order to give

$$\Gamma_1 : \qquad \xi \simeq \tau^+\zeta - \frac{C_1}{\zeta}, \tag{6.22}$$

where $C_1 \geqslant 0$ is an arbitrary constant 'labelling' the characteristics; $C_1 = 0$ corresponds to the leading edge of the undular bore. This asymptotic formula (6.22) is valid as long as $\zeta \gg 1$. The behaviour of the characteristics belonging to the families Γ_1 and Γ_2 near the leading edge is shown in figure 6(*a*).

Next, expanding the equation for the third characteristic family, Γ_3: $\mathrm{d}\xi/\mathrm{d}\zeta = v_3(-1, -m, 0)$ for $(1 - m) \ll 1$, we get, on using (6.20),

$$\frac{\mathrm{d}\xi}{\mathrm{d}\zeta} = \frac{\tau^+ - \xi/\zeta}{\ln(1/(\tau^+ - \xi/\zeta))} + O(\tau^+ - \xi/\zeta). \tag{6.23}$$

Integrating (6.23), we obtain to first order

$$\Gamma_3 : \qquad \xi \simeq C_3 - g(\zeta), \tag{6.24}$$

where

$$g(\zeta) = \int \frac{1}{\zeta} \frac{\tau^+\zeta - C_3}{\ln|\tau^+\zeta - C_3| - \ln\zeta} \, \mathrm{d}\zeta, \qquad g(C_3/\tau^+) = 0, \tag{6.25}$$

C_3 being an arbitrary constant. The asymptotic formula (6.24) is valid as long as $g(\zeta)/C_3 \ll 1$. Since the characteristics Γ_3 intersect the leading edge $\xi = \tau^+\zeta$ we must indicate their behaviour outside the undular bore. It follows from the matching condition (6.8), and the limiting structure (3.25) of the characteristic velocities of the Whitham system, that the characteristics from the family Γ_3 match with the Hopf equation characteristics $\mathrm{d}\xi/\mathrm{d}\zeta = 6r$ carrying the value of the Riemann invariant $r = 0$, corresponding to still water upstream of the undular bore. Therefore, the sought external characteristics are simply vertical lines $\xi = C_3$. The qualitative behaviour of the characteristics from the family Γ_3 is shown in figure 6*b*.

It is clear from the asymptotic behaviour of the characteristics that the edge characteristic $\xi = \tau^+\zeta$ corresponding to the motion of the leading solitary wave intersects only with characteristics of the family Γ_3 carrying the Riemann invariant value $r_3 = 0$ into the undular bore domain. Since, according to the matching condition (6.14), $r_3 \equiv 0$ everywhere along the edge characteristic, one can infer that the leading solitary wave motion is completely specified by its amplitude at $\zeta = 0$. Indeed, in this case, the leading edge represents a genuine multiple characteristic of the modulation system, along which the Riemann invariant $r^+ = r_2 = r_1$ is a constant. Given the constant value of $r_1 = -1$ for the solution (6.16), one infers that the amplitude of the lead soliton of the self-similar undular bore, $\eta_0 = 2(r_3 - r^+) = 2$, is also a constant value. Thus, in the undular bore evolving from an initial jump, the leading solitary wave represents an independent soliton of the KdV equation. Of course, this fact follows directly from the modulation solution (6.16) but now we have established its meaning in the context of the characteristics, which will play an important role below.

Next we discuss the structure of the undular bore front in the case when the initial profile $\eta(\xi, 0)$ is not a simple jump discontinuity, and instead has the form of a monotonically decreasing function, for instance, $(-\xi)^{1/2}$ when $\xi \leqslant 0$ and $\eta(\xi, 0) = 0$ for $\xi > 0$. In that case, the modulation solution for the undular bore no longer possesses x/t-similarity as in the jump resolution case and, as a result, the speed (and therefore, the amplitude) of the lead solitary wave is not constant. For instance, for the aforementioned square-root initial profile the amplitude of the lead solitary wave grows as ζ^2 (see Gurevich Krylov & Mazor 1989, or Kamchatnov 2000). Clearly, such an amplitude variation is impossible if the leading edge $\xi^+(\zeta)$ was a regular characteristic carrying a constant value of the Riemann invariant r^+. As discussed above, however, the GP matching conditions (6.7)–(6.11) admit another possibility: the leading edge curve is the *envelope* of the characteristic families Γ_1: $\mathrm{d}\xi/\mathrm{d}\zeta = v_1$ and Γ_2: $\mathrm{d}\xi/\mathrm{d}\zeta = v_2$ merging when $m = 1$. This configuration is shown in figure 7(*a*). In this case, the behaviour of the modulus m in the vicinity of the leading edge is given by the asymptotic formula found in Gurevich & Pitaevskii (1974):

$$(1-m)^2 \left(\ln \frac{16}{1-m} + \frac{1}{2} \right) = \frac{2}{(r^+)^2} \frac{\mathrm{d}r^+}{\mathrm{d}\zeta} (\xi^+ - \xi), \tag{6.26}$$

where the function $r^+(\zeta) \neq$ constant is assumed to be known. Another specific feature of this (general) configuration is that $\mathrm{d}r_{1,2}/\mathrm{d}\xi \to \pm\infty$ as $\xi \to \xi^+$ (see figure 7*b*, also found in Gurevich & Pitaevskii 1974; see also Kamchatnov 2000), which is in drastic contrast to the similarity solution (see figure 6*a*). This particular difference was

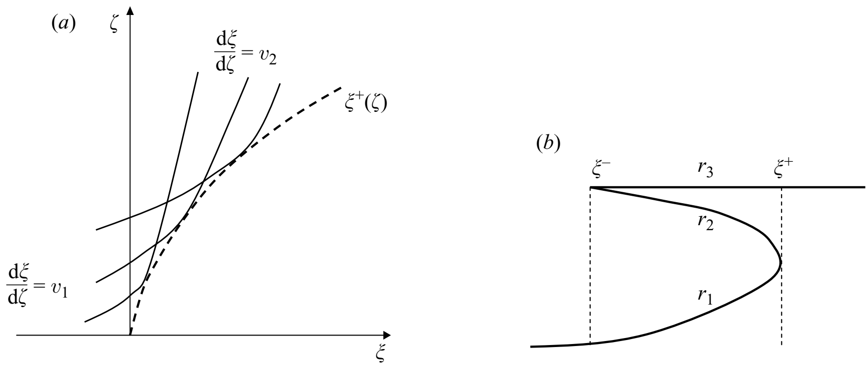

FIGURE 7. (*a*) Leading edge $\xi^+(\zeta)$ of non-self-similar undular bore as an envelope of pairwise-merging characteristics from the families $d\xi/d\zeta = v_1$ and $d\xi/d\zeta = v_2$. (*b*) Behaviour of the Riemann invariants in non-self-similar modulation solution with $r_3 \equiv 0$.

discussed in relation with undular bores in the KdV–Burgers equation in Gurevich & Pitaevskii (1987).

In summary, we see from (6.26) that the structure of the modulation solution in the vicinity of the leading edge of an undular bore defined as a characteristic envelope is qualitatively different compared to that for the similarity case, see (6.19). The more general (but qualitatively similar to (6.26)) asymptotic formula, which takes into account small perturbations due to a variable topography and bottom friction, will be derived later. At the moment, it is important for us that, in this configuration, when the leading edge is a characteristic envelope rather than just a characteristic, the value r^+, and thus the leading solitary wave amplitude, are allowed to vary.

The analysis of the corresponding modulation solution in Gurevich *et al.* (1989) showed that, while in the case of an initial jump the wave crests generated at the trailing edge reach the leading edge (and therefore, transform into solitary waves) only asymptotically as $t \to \infty$, for the more general case of decreasing initial data each wave crest generated at the trailing edge reaches the leading edge in finite time and replaces (overtakes) the existing leading solitary wave. This process is manifested as a continuous amplitude growth of the (apparent) leading solitary wave. As in classical soliton theory, an alternative explanation of the leading solitary wave amplitude growth can be made in terms of the momentum exchange between the 'instantaneous' leading solitary wave and solitary waves of greater amplitude coming from the left. Indeed, as the rigorous analysis of Lax, Levermore & Venakides showed (see Lax, Levermore & Venakides 1994 and the references therein), the whole modulated structure of the undular bore can be asymptotically described in terms of the interactions of a large number of KdV solitons initially 'packed' into a non-oscillating large-scale initial profile.

This latter interpretation is especially instructive for our purposes. Our point is that the specific cause of the enhanced soliton interactions resulting in amplitude growth at the leading edge is not essential; it can be large-scale spatial variations of the initial profile as just described, but it could also equally well be an effect of small perturbations in the KdV equation itself. Indeed, in the weakly perturbed KdV equation, the local wave structure of the undular bore must be described to leading order by the periodic solution (6.4) of the *unperturbed* KdV equation; so if one assumes the GP boundary conditions analogous to (6.7)–(6.11) for the perturbed modulation system (3.18), one will invariably have to deal with one of the two possible types of behaviour of the characteristics (shown in figures 6*a* and 7*a*) in the vicinity of the leading edge of the undular bore, because this qualitative behaviour is determined only by the structure of the GP boundary conditions and by the associated asymptotic structure of the characteristic velocities of the Whitham system for $(1-m) \ll 1$, which are the same for both unperturbed and perturbed modulation systems. Next, we will show that, by using the knowledge of this qualitative behaviour of the characteristics, one is able to construct the asymptotic modulation solution for the undular bore front in the presence of variable topography and bottom friction even if the full solution of the perturbed modulation system is not available.

6.4. *The Gurevich–Pitaevskii problem for the perturbed modulation system*

We now investigate how the GP matching problem applies to the perturbed modulation system (3.18). As in the original GP problem, we postulate the natural physical requirement that the mean value $\langle U \rangle$ is continuous across the undular bore edges, which represent free boundaries and are defined by the conditions $m=0$ (trailing edge $X=X^-(T)$) and $m=1$ (leading edge $X=X^+(T)$). Also, we consider propagation of the undular bore into still water, hence $\langle U \rangle|_{X=X^+(T)}=0$. Now, using the explicit expression (3.9) for $\langle U \rangle$ in terms of complete elliptic integrals and calculating its limits as $m \to 0$ and $m \to 1$, we have

$$\begin{aligned} X = X^-(T): \qquad \lambda_2 = \lambda_3, \quad \langle U \rangle = -\lambda_1 = u, \\ X = X^+(T): \qquad \lambda_2 = \lambda_1, \quad \langle U \rangle = -\lambda_3 = 0, \end{aligned} \tag{6.27}$$

where $u(X, T)$ is solution of the dispersionless perturbed KdV equation (2.2), i.e.

$$u_T + 6uu_X = F(T)u - G(T)u^2, \tag{6.28}$$

with the boundary conditions

$$u\left(\tau, \frac{1}{6g}\int_0^\tau h\,\mathrm{d}\tau\right) = \frac{9g}{2h_0}\Delta_0 \quad \text{if} \quad \tau < \tau_0; \qquad u\left(\tau, \frac{1}{6g}\int_0^\tau h\,\mathrm{d}\tau\right) = 0 \quad \text{if} \quad \tau > \tau_0, \tag{6.29}$$

where $\tau_0 = -x_0/\sqrt{gh_0}$. The boundary conditions (6.29) correspond to a discontinuous initial surface elevation $A(x, t)$ at $x=-x_0$, obtained by using transformations (1.3) and (2.1) where one sets $t=0$. As earlier, $\Delta_0 = a_0/(3h_0)$ is the value of the discontinuity in A, chosen in such a way that the amplitude of the lead solitary wave in the undular bore was exactly a_0 in the flat-bottom zero-friction region (see § 6.2).

This free-boundary matching problem is then complemented by the kinematic conditions explicitly defining the boundaries $X=X^{\pm}(T)$. These are formulated using the multiple characteristic directions of the perturbed modulation system (3.18) in the limits as $m \to 0$ and $m \to 1$ (*cf.* (6.9)–(6.11)),

$$\frac{\mathrm{d}X^-}{\mathrm{d}T} = V^-(X^-, T), \qquad \frac{\mathrm{d}X^+}{\mathrm{d}T} = V^+(X^+, T), \tag{6.30}$$

where

$$V^- = v_2(u, \lambda^-, \lambda^-) = v_3(u, \lambda^-, \lambda^-), \tag{6.31}$$

$$V^+ = v_2(\lambda^+, \lambda^+, 0) = v_1(\lambda^+, \lambda^+, 0), \tag{6.32}$$

$$\lambda^- = \lambda_2(X^-, T) = \lambda_3(X^-, T), \qquad \lambda^+ = \lambda_2(X^+, T) = \lambda_1(X^+, T). \tag{6.33}$$

Thus, for the perturbed KdV equation the leading and trailing edges of the undular bore are defined mathematically in the same way as for the unperturbed one, albeit for a different set of variables.

6.5. *Deformation of the undular bore front due to variable topography and bottom friction*

Finally we study the effects of gradual slope and bottom friction on the leading front of the self-similar expanding undular bore described in § 6.2 and § 6.3. The result will essentially depend on the relative values of the small parameters appearing in the problem. We note that in general there are three distinct relevant small parameters,

$$\epsilon = \frac{h_0}{x_0} \ll 1, \qquad \delta = \max(h_x) \ll 1, \qquad C_D \ll 1. \tag{6.34}$$

The first small parameter is determined by the ratio of the equilibrium depth in the flat bottom region, to the distance from the beginning of the slope region to the location of the initial jump discontinuity in the surface displacement. This measures the typical relative spatial variations of the modulation parameters in the undular bore when it reaches the beginning of the slope. The second and third parameters are contained in the KdV equation (1.1) itself, and measure the values of the slope and bottom friction respectively. In terms of the transformed variables appearing in (2.2), $|F(T)| \sim \delta$, $|G(T)| \sim C_D$; see (2.3). Generally we assume $\delta \sim C_D$ (the possible orderings $\delta \ll C_D$ or $C_D \ll \delta$ can be then considered as particular cases).

To obtain a quantitative description of the vicinity of the leading edge of the undular bore we perform an expansion of the Whitham modulation system (3.18) for $(1-m) \ll 1$. We first introduce the substitutions

$$\lambda_i(X,T) = \lambda^+(T) + l_i(\tilde{X},T), \quad v_i = V^+ + v_i', \quad \rho_i = \rho^+ + \rho_i', \quad i = 1, 2. \tag{6.35}$$

where

$$\tilde{X} = X^+ - X, \qquad V^+ = -4\lambda^+, \quad \rho^+ = \frac{4}{3}F(T)\lambda^+ + \frac{32}{15}G(T)(\lambda^+)^2. \tag{6.36}$$

Since $\lambda_2 \geqslant \lambda_1$, $v_2 \geqslant v_1$ one always has $l_2 \geqslant l_1$, $v_2' \geqslant v_1'$. Assuming $\tilde{X}/X^+ \ll 1 \Leftrightarrow 1-m \ll 1$ and using that $\lambda_3 = 0$ to leading order in the vicinity of the leading edge (see the matching condition (6.27)), we have from asymptotic expansions of (3.19)–(3.22) as $(1-m) \ll 1$:

$$\left.\begin{aligned} v_1' = M_1(l_2-l_1) \equiv -2\left[1 + \frac{\ln(16/(1-m))}{1+\frac{1}{4}(1-m)\ln(16/(1-m))}\right](l_2-l_1), \\ v_2' = M_2(l_2-l_1) \equiv -2\left[1 - \frac{\ln(16/(1-m))}{1-\frac{1}{4}(1-m)\ln(16/(1-m))}\right](l_2-l_1), \end{aligned}\right\} \tag{6.37}$$

$$\left.\begin{aligned} \rho_1' = N_1(l_2-l_1) \equiv & \left\{-\frac{1}{3}\left[1 + \ln\frac{l_2-l_1}{-16\lambda^+}\right]F \right. \\ & \left. -\frac{4}{15}\left[2\lambda^+\ln\frac{l_2-l_1}{-16\lambda^+} - 3\lambda^+\right]G\right\}(l_2-l_1), \\ \rho_2' = N_2(l_2-l_1) \equiv & \left\{\frac{1}{3}\left[5 + \ln\frac{l_2-l_1}{-16\lambda^+}\right]F \right. \\ & \left. +\frac{4}{15}\left[2\lambda^+\ln\frac{l_2-l_1}{-16\lambda^+} + 13\lambda^+\right]G\right\}(l_2-l_1). \end{aligned}\right\} \tag{6.38}$$

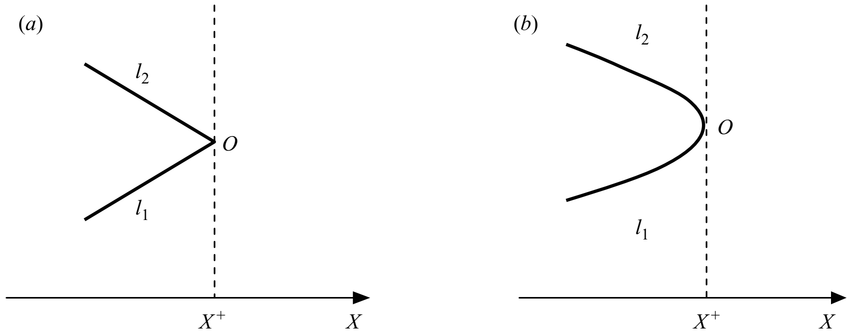

FIGURE 8. Behaviour of Riemann variables in the vicinity of the leading edge of the undular bore propagating over gradual slope with bottom friction. (*a*) Adiabatic variations of the similarity GP regime, $\delta \ll \epsilon$, $C_D \ll \epsilon$. (*b*) General case, $\delta \sim C_D \sim \epsilon$.

Naturally, v_i' and ρ_i' vanish when $l_2 = l_1$. Now, substituting (6.35), (6.36) into the modulation system (3.18), we obtain

$$\frac{d\lambda^+}{dT} + \frac{\partial l_i}{\partial \tilde{X}}\frac{dX^+}{dT} - (V^+ + v_i')\frac{\partial l_i}{\partial \tilde{X}} = \rho^+ + \rho_i', \quad i = 1, 2. \tag{6.39}$$

On using the kinematic condition (6.30) at the leading edge, this reduces to

$$\frac{d\lambda^+}{dT} - v_i'\frac{\partial l_i}{\partial \tilde{X}} = \rho^+ + \rho_i', \quad i = 1, 2. \tag{6.40}$$

There are two qualitatively different cases to consider:

(i) $\lim_{\tilde{X}\to 0} |dl_i/d\tilde{X}| < \infty$, $i = 1, 2$ (figure 8*a*),

(ii) $\lim_{\tilde{X}\to 0} |dl_i/d\tilde{X}| = \infty$, $i = 1, 2$ (figure 8*b*).

The case (i) implies that, to leading order, (6.40) reduces to

$$\frac{d\lambda^+}{dT} = \rho^+, \tag{6.41}$$

which, together with the kinematic condition $dX^+/dT = -4\lambda^+$, defines the leading edge curve $X^+(T)$. One can observe that this system coincides with (4.6), (4.5), defining the motion of a separate solitary wave over a gradual slope with bottom friction. Its integral expressed in terms of original physical x, t-variables is given by (4.12). Therefore, in the case(i), the lead solitary wave in the undular bore to leading order is not restrained by interactions with the remaining part of the bore and behaves as a separate solitary wave. Physically this case corresponds to adiabatic deformation of the similarity modulation solution (6.15), (6.16) and implies the following small parameter ordering: $\delta \ll \epsilon$, $C_D \ll \epsilon$.

Next, we study the structure of this weakly perturbed similarity modulation solution in the vicinity of the leading edge. The next leading order of the system (6.40) yields

$$-v_i'\frac{\partial l_i}{\partial \tilde{X}} = \rho_i', \quad i = 1, 2, \tag{6.42}$$

that is,

$$\frac{\partial l_1}{\partial \tilde{X}} = -\frac{N_1}{M_1}, \qquad \frac{\partial l_2}{\partial \tilde{X}} = -\frac{N_2}{M_2}. \tag{6.43}$$

Subtraction of one equation (6.43) from another, taking account of the relationship $l_2 - l_1 \cong -\lambda^+(1-m)$, leads consistently, to leading order, to the differential equation for $1-m$:

$$\frac{\partial(1-m)}{\partial\tilde{X}} = 2\left[\frac{F(T)}{-3\lambda^+} - \frac{16G(T)}{15}\right]\left(\ln\frac{16}{1-m}\right)^{-1}. \tag{6.44}$$

This equation should be solved with the initial condition

$$1-m=0 \quad \text{at} \quad \tilde{X}=0. \tag{6.45}$$

Elementary integration gives, with accuracy $O(1-m)$ (*cf.* (6.19)),

$$(1-m)\ln\frac{16}{1-m} = -2\left[\frac{1}{3}F(T) - \frac{16}{15}\lambda^+G(T)\right]\frac{X^+-X}{-\lambda^+}. \tag{6.46}$$

This formula determines the dependence of the modulus m on T and X (as long as $1-m \ll 1$).

Now, we make use of the solution λ^+ of equation (6.41) given by (4.10) with $C_0 = 4/(3ga_0h_0)$ (see (4.11)). Under the assumption that the integral $\int^x h^{-3}\,\mathrm{d}x$ diverges as $h \to 0$, so that turbulent bottom friction plays an essential role in the undular bore front behaviour (see § 4 for a similar approximation for an isolated solitary wave), we obtain for $h \ll h_0$

$$(1-m)\ln\frac{16}{1-m} = \frac{64}{15}C_D\left(2+3h^2\int_0^x\frac{\mathrm{d}x}{h^3}\right)(X^+-X). \tag{6.47}$$

Finally, if the bottom topography is approximated by the dependence (4.15), we get with the same accuracy

$$(1-m)\ln\frac{16}{1-m} = \frac{64}{15}C_D\left[2+\frac{3}{(3\alpha-1)\delta}\left(\frac{h}{h_0}\right)^{1/\alpha}\right](X^+-X), \tag{6.48}$$

where $\alpha > 1/3$. The second term in square brackets tends to zero as $h \to 0$. However, the region where it can be neglected may be very narrow because of smallness of the parameter δ. We recall that in this formula X^+ is given by (4.12) and X is defined by (1.3) in terms of the original physical independent variables x and t.

Summarizing, if the conditions $\delta, C_D \ll \epsilon$ are satisfied, the lead solitary wave of the undular bore behaves as an individual (non-interacting) solitary wave adiabatically varying under small perturbation due to variable topography and bottom friction. The modulation solution in the vicinity of the leading edge also varies adiabatically; however, its *qualitative* structure considered in § 6.4 (see figures 5 and 6) remains unchanged.

In sharp contrast to the described case of adiabatic deformation of an undular bore front is case (ii), when the second term on the left-hand side of (6.40) contributes to the leading order, i.e. to the motion of the leading edge itself. Namely, we have

$$\frac{\mathrm{d}\lambda^+}{\mathrm{d}T} = \rho^+ + v_i'\frac{\partial l_i}{\partial\tilde{X}}, \quad i=1,2. \tag{6.49}$$

Now $\mathrm{d}\lambda^+/\mathrm{d}T \neq \rho^+$, which means that the amplitude of the lead solitary wave $a=-2\lambda^+$ varies essentially differently compared to the case of an isolated solitary wave. Indeed, the term ρ^+ on the right-hand side of (6.49) is responsible for local adiabatic variations of the solitary wave, while the term $v_i'\partial l_i/\partial\tilde{X}$ describes non-local parts of the variations associated with the wave interactions within the undular bore. Using asymptotic formulae (6.37) implying $v_2' \geqslant 0$, $v_1' \leqslant 0$, and the condition $\lim_{\tilde{X}\to 0}|\mathrm{d}l_{1,2}/\mathrm{d}\tilde{X}| = \infty$ along with $l_2 \geqslant l_1$, it is not difficult to show that this non-local term is always non-negative, i.e. the lead solitary wave in the undular bore propagating over a gradual slope with bottom friction always moves faster (and therefore has greater amplitude) than an isolated solitary wave of the same initial amplitude in the beginning of the slope. Indeed, as we have shown in § 5, the presence of a slope and bottom friction always result in 'squeezing' the cnoidal wave, hence increasing momentum exchange between solitary waves in the vicinity of the leading edge of the undular bore and acceleration of the lead solitary wave itself. The situation here is qualitatively analogous to that described in § 6.4, where the general global modulation solution for the unperturbed KdV equation was discussed. As in that case, the leading edge now represents a characteristic envelope – a caustic (otherwise we are back in the case (i) implying $d\lambda^+/dT = \rho^+$) (see figure 6*a*).

Unlike the case of adiabatic variations of the leading edge, determination of the function $\lambda^+(T)$ now requires knowledge of the full solution of the perturbed modulation system (3.18) with the matching conditions (6.27). While the analytic methods to construct such a solution for inhomogeneous quasilinear systems are not at present available, it is instructive to assume that $d\lambda^+/dT - \rho^+$ is a known function of T and to study the structure of the solution in close vicinity of the leading edge. With an account of the explicit form (6.37) of the velocity corrections, equations (6.49) assume the form

$$\frac{\partial l_2}{\partial \tilde{X}} = -\frac{d\lambda^+/dT - \rho^+}{2(l_2 - l_1)} \left[\frac{1}{\ln[16/(1-m)]} + \frac{1}{4}(1-m) \right], \tag{6.50}$$

$$\frac{\partial l_1}{\partial \tilde{X}} = -\frac{d\lambda^+/dT - \rho^+}{2(l_2 - l_1)} \left[-\frac{1}{\ln[16/(1-m)]} + \frac{1}{4}(1-m) \right]. \tag{6.51}$$

Taking the difference of (6.50) and (6.51), we transform it into the form

$$\frac{\partial(1-m)}{\partial X} = \frac{d\lambda^+/dT - \rho^+}{(\lambda^+)^2} \frac{1}{(1-m)\ln[16/(1-m)]}. \tag{6.52}$$

This equation can be readily integrated with the initial condition (6.45) to give

$$(1-m)^2 \left(\ln \frac{16}{1-m} + \frac{1}{2} \right) = \frac{2(d\lambda^+/dT - \rho^+)}{(\lambda^+)^2} (X^+ - X). \tag{6.53}$$

This solution coincides with the asymptotic formula (6.26) for the behaviour of the modulus in the vicinity of the leading edge of the undular bore in general unperturbed GP problem (Gurevich & Pitaevskii 1974), but instead of the derivative $d\lambda^+/dT$ in (6.26) we have the difference $d\lambda^+/dT - \rho^+$ (which is always positive, as we have established).

7. Conclusions

We have studied the effects of a gradual slope and turbulent (Chezy) bottom friction on the propagation of solitary waves, nonlinear periodic waves and undular bores in shallow-water flows in the framework of the variable-coefficient perturbed KdV equation. The analysis has been performed in the most general setting provided by the associated Whitham equations, describing slow modulations of a periodic travelling wave due to the slope, bottom friction and spatial non-uniformity of initial data. This modulation theory, developed in general form for perturbed integrable equations in Kamchatnov (2004), was applied here to the perturbed KdV equation and allowed us to take into account slow variations of all three parameters in the cnoidal wave solution. The particular time-independent solutions of the perturbed modulation equations were shown to be consistent with the adiabatically varying solutions for a single solitary wave and for a periodic wave propagating over a slope without bottom friction obtained in Ostrovsky & Pelinovsky (1970, 1975) and Miles (1979, 1983*a*). It was shown, however, that the assumption of zero mean elevation used in these papers for the description of slow variations of a cnoidal wave, ceases to be valid in the case when the turbulent bottom friction is present. In this case, a more general solution was obtained numerically, improving the results of Miles (1983*b*).

Further, the derived full time-dependent modulation system was used for the description of the effects of variable topography and bottom friction on the propagation of undular bores, in particular on the variations of the undular bore front representing a system of weakly interacting solitary waves. By the analysis of the characteristics of the Whitham system in the vicinity of the leading edge of the undular bore, two possible configurations have been identified, depending on whether the leading edge of the undular bore represents a regular characteristic of the modulation system or its singular characteristic, i.e. a caustic. The first case was shown to correspond to adiabatically slow deformations of the classical Gurevich–Pitaevskii modulation solution, and is realized when the perturbations due to variable topography and bottom friction are small compared with the existing spatial non-uniformity of modulations in the undular bore (which is supposed to be formed outside the region of variable topography/bottom friction). In the case when modulations due to the external perturbations are comparable in magnitude with the existing modulations in the undular bore, the leading edge becomes a caustic, and this situation was shown to correspond to enhanced solitary wave interactions within the undular bore front. These enhanced interactions have been shown to lead to a 'non-local' leading solitary wave amplitude growth, which cannot be predicted in the frame of the traditional local adiabatic approach to propagation of an isolated solitary wave in a variable environment. As we mentioned in the Introduction, one of our original motivations for this study was the possibility of modelling a shoreward propagating tsunami as an undular bore. In this context, we would suggest that the second scenario described above is the more relevant, which has the implication that the growth and eventual breaking of the leading waves in a tsunami wavetrain cannot be modelled as a local effect for that particular wave, but is determined instead by the whole structure of the wavetrain.

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Appendix. Derivation of the perturbed modulation system

We express the integrand function on the right-hand side of (3.15) in terms of the μ-variable (3.6):

$$(2\lambda_i - s_1 - U)R = 8G\mu^3 - [8G\lambda_i + 4(F + 2s_1G)]\mu^2 + [4(F + 2s_1G)\lambda_i + 2s_1(s_1G + F)]\mu - 2s_1(s_1G + F)\lambda_i. \quad \text{(A 1)}$$

Then we obtain with the use of (3.4), (3.5) and (3.7) the following expressions:

$$\left.\begin{aligned}
\langle\mu\rangle &= \frac{1}{L}\oint \mu \mathrm{d}\theta = \frac{1}{L}\oint \mu\frac{\mathrm{d}\theta}{\mathrm{d}\mu}\mathrm{d}\mu = \frac{1}{L}\oint \frac{\mu \mathrm{d}\mu}{2\sqrt{-P(\mu)}} = -\frac{2}{L}\frac{\partial I}{\partial s_2},\\
\langle\mu^2\rangle &= \frac{1}{L}\oint \mu^2 \mathrm{d}\theta = \frac{2}{L}\frac{\partial I}{\partial s_1}\\
\langle\mu^3\rangle &= \frac{1}{L}\oint \mu^3 d\theta = -\frac{I}{L} + s_1\langle\mu^2\rangle - s_2\langle\mu\rangle + s_3,
\end{aligned}\right\} \tag{A 2}$$

where I is a known integral

$$\begin{aligned}
I &= \int_{\lambda_2}^{\lambda_3} \sqrt{(\lambda_3-\mu)(\mu-\lambda_2)(\mu-\lambda_1)}\, d\mu\\
&= \frac{4}{15}(\lambda_3-\lambda_1)^{5/2}[(1-m+m^2)E(m)-(1-m)(1-m/2)K(m)],
\end{aligned} \tag{A 3}$$

$K(m)$ and $E(m)$ being the complete elliptic integrals of the first and second kind, respectively. The derivatives of I with respect to λ_i are also known table integrals (Gradshtein & Ryzhik 1980):

$$\left.\begin{aligned}
\frac{\partial I}{\partial \lambda_1} &= -\frac{1}{2}\int_{\lambda_2}^{\lambda_3}\sqrt{\frac{(\lambda_3-\mu)(\mu-\lambda_2)}{\mu-\lambda_1}}\mathrm{d}\mu\\
&= -\frac{1}{3}\sqrt{\lambda_3-\lambda_1}[(\lambda_2+\lambda_3-2\lambda_1)E - 2(\lambda_2-\lambda_1)K],\\
\frac{\partial I}{\partial \lambda_2} &= -\frac{1}{2}\int_{\lambda_2}^{\lambda_3}\sqrt{\frac{(\lambda_3-\mu)(\mu-\lambda_1)}{\mu-\lambda_2}}\mathrm{d}\mu\\
&= -\frac{1}{3}\sqrt{\lambda_3-\lambda_1}[(\lambda_3-\lambda_1)K + (\lambda_1+\lambda_3-2\lambda_2)E],\\
\frac{\partial I}{\partial \lambda_3} &= \frac{1}{2}\int_{\lambda_2}^{\lambda_3}\sqrt{\frac{(\mu-\lambda_2)(\mu-\lambda_1)}{\lambda_3-\mu}}\mathrm{d}\mu\\
&= \frac{1}{3}\sqrt{\lambda_3-\lambda_1}[(2\lambda_3-\lambda_1-\lambda_2)E - (\lambda_2-\lambda_1)K].
\end{aligned}\right\} \tag{A 4}$$

We can easily express the s_i-derivatives in terms of λ_i-derivatives by differentiation of the formulae (see (3.7))

$$s_1 = \lambda_1+\lambda_2+\lambda_3, \quad s_2 = \lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3, \quad s_3 = \lambda_1\lambda_2\lambda_3 \tag{A 5}$$

and solving the linear system for differentials. Simple calculation gives

$$\frac{\partial \lambda_i}{\partial s_k} = \frac{(-1)^{3-k}}{\prod_{j\neq i}(\lambda_i-\lambda_j)}. \tag{A 6}$$

Then, combining (A 4) and (A 6), we obtain the derivatives $\partial I/\partial s_i$ and hence the expressions

$$\left.\begin{aligned} \frac{I}{L} &= \frac{2}{15}(\lambda_3-\lambda_1)\left[\left(s_1^2-3s_2\right)\frac{E}{K}-\frac{1}{2}(\lambda_2-\lambda_1)(\lambda_2+\lambda_3-2\lambda_1)\right], \\ \frac{1}{L}\frac{\partial I}{\partial s_1} &= \frac{1}{6}\left[2s_1\frac{E}{K}+s_1\lambda_1+\lambda_1^2-\lambda_2\lambda_3\right], \\ \frac{1}{L}\frac{\partial I}{\partial s_2} &= -\frac{1}{2}\left[(\lambda_3-\lambda_1)\frac{E}{K}+\lambda_1\right]. \end{aligned}\right\} \tag{A 7}$$

To complete the calculation of the right-hand side of (3.15), we also need expressions

$$\left.\begin{aligned} \frac{L}{\partial L/\partial\lambda_1} &= 2(\lambda_2-\lambda_1)\frac{K}{E}, \\ \frac{L}{\partial L/\partial\lambda_2} &= -\frac{2(\lambda_3-\lambda_2)(1-m)K}{E-(1-m)K}, \\ \frac{L}{\partial L/\partial\lambda_3} &= \frac{2(\lambda_3-\lambda_2)K}{E-K}. \end{aligned}\right\} \tag{A 8}$$

Collecting all contributions into perturbation terms, we obtain the Whitham equations in the form

$$\frac{\partial\lambda_i}{\partial T}+v_i\frac{\partial\lambda_i}{\partial X}=\rho_i=C_i[F(T)A_i-G(T)B_i], \tag{A 9}$$

where C_j, A_j, B_j and v_j, $j=1, 2, 3$ are specified by formulae (3.19)–(3.21).

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